UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in MAT2400 — Analysis 1.

Day of examination: Wednesday, June 15th, 2011.

Examination hours: 09.00 – 13.00

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All items (1, 2, 3a, 3b etc.) count 10 points.

Problem 1

The functions $f_n : \mathbb{R} \to \mathbb{R}$ are defined by

$$f_n(x) = \arctan(x^{2n})$$

Show that

$$\lim_{n\to\infty} f_n(x) = \begin{cases} \frac{\pi}{2} & \text{if } |x| > 1 \\ \frac{\pi}{4} & \text{if } |x| = 1 \\ 0 & \text{if } |x| < 1 \end{cases}$$

Is the convergence uniform?

Problem 2

(X, d) is a metric space, and A is a subset of X. We say that $b \in X$ is an *accumulation point* for A if any ball $B(b; r)$, $r > 0$, contains a point from A different from b. Show that b is an accumulation point for A if and only if there is a sequence $\{x_n\}$ from A which converges to b and where all the elements are different from b.

Problem 3

In this problem, μ is the Lebesgue measure on $\mathbb{R}^d$, and $\mathcal{M}^+$ is the set of non-negative, measurable functions $f : \mathbb{R}^d \to [0, \infty]$. Assume that $I : \mathcal{M}^+ \to [0, \infty]$ satisfies the following four conditions:

(i) $I(cf) = cI(f)$ for all $c \in [0, \infty)$ and all $f \in \mathcal{M}^+$.

(ii) $I(f + g) = I(f) + I(g)$ for all $f, g \in \mathcal{M}^+$.

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(iii) $I(\mathbf{1}_A) = \mu(A)$ for all measurable $A \subset \mathbb{R}^d$.

(iv) If $\{f_n\}$ is an increasing sequence from $\mathcal{M}^+$ converging to f, then $\lim_{n\to\infty} I(f_n) = I(f)$.

a) Show by induction that

$$I(c_1 f_1 + c_2 f_2 + \cdots + c_n f_n) = c_1 I(f_1) + c_2 I(f_2) + \cdots + c_n I(f_n)$$

for all $n \in \mathbb{N}$, all $c_1, c_2, \ldots, c_n \in [0, \infty)$ and all $f_1, f_2, \ldots, f_n \in \mathcal{M}^+$.

b) Show that $I(f) = \int f \, d\mu$ for all non-negative, simple functions f.

c) Show that $I(f) = \int f \, d\mu$ for all $f \in \mathcal{M}^+$.

Problem 4

In this problem X is the set of all functions

$$f : \mathbb{N} \to \mathbb{R}$$

such that

$$\sum_{i=1}^{\infty} |f(i)| < \infty$$

a) Show that

(i) If $f \in X$ and $c \in \mathbb{R}$, then $cf \in X$.
(ii) If $f, g \in X$, then $f + g \in X$.

The result in part a) tells us that X is a vector space. You may use this freely in what follows.

b) Show that

$$\|f\| = \sum_{i=1}^{\infty} |f(i)|$$

is a norm on X.

c) For all $n \in \mathbb{N}$, we define $e_n : \mathbb{N} \to \mathbb{R}$ by

$$e_n(i) = \begin{cases} 1 & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

Show that $\{e_n\}_{n\in\mathbb{N}}$ is a basis for X.

d) Show that $\lim_{n\to\infty} \|f - e_n\| = \|f\| + 1$ for all $f \in X$. Why can't $\{e_n\}$ have a convergent subsequence in $(X, \|\cdot\|)$?

e) Show that

$$B = \{f \in X : \|f\| \leq 1\}$$

is closed and bounded, but not compact. Is B totally bounded?

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Problem 5

In this problem (X, d) is a metric space, A is a subset of X, $f : A \to \mathbb{R}$ is a uniformly continuous function, and b is a boundary point of A. Show that if $\{x_n\}$ is a sequence from A converging to b, then the sequence $\{f(x_n)\}$ converges. Find an example which shows that this does not always hold if we only assume that f is continuous.

THE END