UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT2400 — Real analysis

Day of examination: Tuesday, June 4th, 2013

Examination hours: 14.30 – 18.30

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All items (Problems 1, 2, 3a, 3b, osv) count 10 points each.

Problem 1: Show that the sequence $f_n(x) = nxe^{-nx^2}$ converges pointwise on the interval $[0, 1]$ and decide whether the convergence is uniform.

Problem 2: Assume that (X, d) is a metric space and that $\{f_n\}$ and $\{g_n\}$ are two sequences of functions from X to $\mathbb{R}$ which converge uniformly to f and g, respectively. Show that $\{f_n + g_n\}$ converges uniformly to $f + g$.

Problem 3: In this problem μ is the Lebesgue measure on $\mathbb{R}$, and $\mathcal{A}$ is the σ-algebra of Lebesgue measurable sets. We write $L^1(\mu)$ as an abbreviation of $L^1(\mathbb{R}, \mathcal{A}, \mu)$, and we let $\|\cdot\|_1$ denote the L^1-norm (i.e. $\|f\|_1 = \int |f|\, d\mu$).

a) Show that for each $f \in L^1(\mu)$ there is a sequence $\{g_n\}$ of simple functions converging to f in $L^1(\mu)$ (i.e. $\|f - g_n\|_1 \to 0$ as $n \to \infty$).

For each $A \in \mathcal{A}$ and each $\epsilon > 0$ there is a continuous function $h : \mathbb{R} \to [0, 1]$ such that $\mu(\{x \in \mathbb{R} \,|\, h(x) \neq \mathbf{1}_A(x)\}) < \epsilon$. You may use this freely (and without proving it) in the rest of the problem.

b) Show that $\|\mathbf{1}_A - h\|_1 < \epsilon$ when A and h are as above.

c) Show that for every simple function $g \in L^1(\mu)$ and every $\epsilon > 0$ there is a continuous function h such that $\|g - h\|_1 < \epsilon$.

d) Show that for every $f \in L^1(\mu)$ there is a sequence $\{h_n\}$ of continuous functions such that $\|f - h_n\|_1 \to 0$.

Problem 4: In this problem $(H, \langle\cdot,\cdot\rangle)$ is a real Hilbert space (i.e. a complete inner product space over $\mathbb{R}$) with orthonormal basis $\{\mathbf{e}_i\}_{i\in\mathbb{N}}$.

a) Let $\{\mathbf{u}_n\}$ and $\{\mathbf{v}_n\}$ be two sequences in H. Show that if $\lim_{n\to\infty} \mathbf{u}_n = \mathbf{u}$ and $\lim_{n\to\infty} \mathbf{v}_n = \mathbf{v}$, then $\lim_{n\to\infty}\langle \mathbf{u}_n, \mathbf{v}_n\rangle = \langle \mathbf{u}, \mathbf{v}\rangle$.

b) Show that if $\mathbf{u} = \sum_{i=1}^{\infty} \alpha_i \mathbf{e}_i$ and $\mathbf{v} = \sum_{i=1}^{\infty} \beta_i \mathbf{e}_i$ are two elements in H, then $\langle \mathbf{u}, \mathbf{v}\rangle = \sum_{i=1}^{\infty} \alpha_i\beta_i$.

(Continued on page 2.)

A *linear functional* on H is a function $A : H \to \mathbb{R}$ such that

(i) $A(\alpha\mathbf{u}) = \alpha A(\mathbf{u})$ for all $\alpha \in \mathbb{R}$ and $\mathbf{u} \in H$.

(ii) $A(\mathbf{u} + \mathbf{v}) = A(\mathbf{u}) + A(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in H$.

In the rest of the problem A is a linear functional.

c) Show that

(I) $A(\sum_{i=1}^n \alpha_i \mathbf{u}_i) = \sum_{i=1}^n \alpha_i A(\mathbf{u}_i)$ for all $n \in \mathbb{N}$, all $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and all $\mathbf{u}_1, \ldots, \mathbf{u}_n \in H$

(II) $A(\mathbf{u} - \mathbf{v}) = A(\mathbf{u}) - A(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in H$.

From now on we assume that there is a number $M \in \mathbb{R}$ such that $|A(\mathbf{u})| \leq M\|\mathbf{u}\|$ for all $\mathbf{u} \in H$. (As usual, $\|\cdot\|$ by is the norm generated by the inner product $\langle \cdot, \cdot \rangle$, i.e. $\|\mathbf{u}\| = (\langle \mathbf{u}, \mathbf{u} \rangle)^{\frac{1}{2}}$.)

d) Show that A is uniformly continuous.

e) Let $\beta_i = A(\mathbf{e}_i)$ and show that $A(\sum_{i=1}^n \beta_i \mathbf{e}_i) = \sum_{i=1}^n \beta_i^2$ for all $n \in \mathbb{N}$. Use this to show that $\left(\sum_{i=1}^\infty \beta_i^2\right)^{\frac{1}{2}} \leq M$.

f) Show that the series $\sum_{i=1}^\infty \beta_i \mathbf{e}_i$ converges in H.

g) Let $\mathbf{y} = \sum_{i=1}^\infty \beta_i \mathbf{e}_i$. Show that $A(\mathbf{x}) = \langle \mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x} \in H$.

THE END