UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT2400 — Real analysis

Day of examination: Tuesday, June 2nd, 2014

Examination hours: 09.00 – 13.00

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All items (Problems 1, 2, 3, 4a, 4b, etc) count 10 points

Problem 1: The functions $f_n : [0, \infty) \to \mathbb{R}$ are given by

$$f_n(x) = n^2 x^2 e^{-nx}$$

a) Show that the sequence $\{f_n\}$ converges pointwise.

b) Does the sequence converge uniformly?

Problem 2: In this problem (X_1, d_1) and (X_2, d_2) are two metric spaces. Assume that (X_1, d_1) and (X_2, d_2) are isometric, i.e., there exists a bijection $\phi : X_1 \to X_2$ such that

$$d_1(x, y) = d_2(\phi(x), \phi(y)) \quad \text{for all } x, y \in X_1.$$

Show that if (X_1, d_1) is complete, then (X_2, d_2) is also complete.

Problem 3: Assume that $(X, \mathcal{A}, \mu)$ is a measure space. Recall that a sequence $\{f_n\}$ of functions from X to $\mathbb{R}$ is called *bounded* if there is an $M \in \mathbb{R}$ such that $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in X$. Assume that $\{f_n\}$ is a bounded sequence of measurable functions converging pointwise to a function f. Show that if $\mu(X) < \infty$, then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$$

Find an example which shows that this is not always the case when $\mu(X) = \infty$.

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Problem 4: In this problem X is a nonempty set and $\mathcal{D}$ is a nonempty family of subsets of X such that the following three conditions are satisfied:

(i) If $A \in \mathcal{D}$, then $A^c \in \mathcal{D}$.

(ii) If $A, B \in \mathcal{D}$, then $A \cap B \in \mathcal{D}$.

(iii) If $\{A_n\}_{n\in\mathbb{N}}$ is a disjoint sequence of sets in $\mathcal{D}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{D}$.

a) Show that $\emptyset \in \mathcal{D}$.

b) Show by induction that if $A_1, A_2, \ldots, A_n \in \mathcal{D}$, then $A_1 \cap A_2 \cap \ldots \cap A_n \in \mathcal{D}$.

c) Show that if $A_1, A_2, \ldots, A_n \in \mathcal{D}$, then $A_1 \cup A_2 \cup \ldots \cup A_n \in \mathcal{D}$.

d) Show that $\mathcal{D}$ is a σ-algebra.

Problem 5: In this problem, d_1 and d_2 are two metrics on the same nonempty set X. Define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = d_1(x, y) + d_2(x, y) \quad \text{for all } x, y \in X.$$

a) Show that d is a metric on X.

We say that d_1 and d_2 are *compatible* if the following condition is satisfied:

Condition: If a sequence $\{x_n\}$ converges to a in the d_1-metric and to b in the d_2-metric, then $a = b$.

b) Assume that d_1 and d_2 are compatible. Show that if $C \subset X$ is compact with respect to d_1 and d_2, then C is also compact with respect to d.

Problem 6: Assume that $\{x_n\}$ is a sequence of real numbers converging to $a \in \mathbb{R}$. Show that $\{x_n\}$ is bounded, i.e., that there exists an $M \in \mathbb{R}$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Show also that if

$$y_n = \frac{x_1 + x_2 + \cdot + x_n}{n} \quad \text{for all } n \in \mathbb{N},$$

then the sequence $\{y_n\}$ converges to a.

THE END