23/5 - 2014

MAT 2400

Survey I

Two main themes in the course:

(i) metric spaces - measuring distance

(ii) measure spaces - measuring size

<u>Definition of metric space:</u> (X, d) where d satisfies:

(i) <u>Positivity</u>: $d(x,y) \geq 0$ with equality if and only if $x = y$

(ii) <u>Symmetry</u>: $d(x,y) = d(y,x)$ for all $x, y \in X$

(iii) <u>Triangle inequality</u>: $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

<u>Inverse triangle inequality</u>: $|d(x,y) - d(x,z)| \leq d(y,z)$

<u>Structures that induce metrics:</u>

<u>1</u> Normed spaces $(X, \|\cdot\|)$: X vector space over $\mathbb{K}$

(i) $\|\vec{x}\| \geq 0$ with equality if and only if $\vec{x} = \vec{0}$

(ii) $\|\alpha \vec{x}\| = |\alpha| \|\vec{x}\|$ for all $\alpha \in \mathbb{K}$, $\vec{x} \in X$

(iii) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ for all $\vec{x}, \vec{y} \in X$

The induced metric is $d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\|$

2 Inner product spaces: $(X, \langle\cdot,\cdot\rangle)$ X vector space over $\mathbb{K}$, the inner product $\langle\cdot,\cdot\rangle$ satisfies

(i) $\langle\vec{u},\vec{v}\rangle = \overline{\langle\vec{v},\vec{u}\rangle}$

(ii) $\langle\alpha\vec{u},\vec{v}\rangle = \alpha\langle\vec{u},\vec{v}\rangle$

(iii) $\langle\vec{u},\vec{v}+\vec{w}\rangle = \langle\vec{u},\vec{v}\rangle + \langle\vec{u},\vec{w}\rangle$

(iv) $\langle\vec{u},\vec{u}\rangle \geq 0$ with equality if and only if $\vec{u}=\vec{0}$

The inner product $\langle\cdot,\cdot\rangle$ induces a norm by $\|\vec{u}\| = \langle\vec{u},\vec{u}\rangle^{1/2}$ and a metric by

$$d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\| = \langle\vec{u}-\vec{v},\vec{u}-\vec{v}\rangle^{1/2}$$

Basic concepts in metric spaces

Convergence: $\{x_n\}$ converges to a if for all $\varepsilon>0$ there is an $N\in\mathbb{N}$ such that $d(x_n,a)<\varepsilon$ for all $n\geq N$.

Equivalently: $d(x_n,a)\to 0$ as $n\to\infty$
(note that $\{d(x_n,a)\}$ is a sequence of numbers)

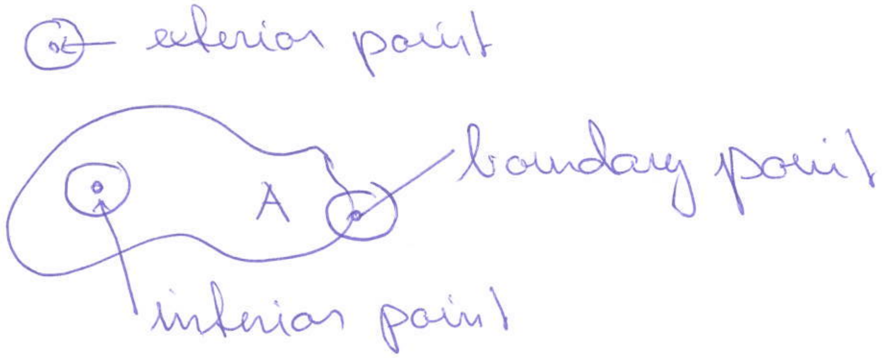
Continuity: $f: X\to Y$ is continuous at a if for all $\varepsilon>0$, there is a $\delta>0$ such that $d_Y(f(x),f(a))<\varepsilon$ whenever $d_X(x,a)<\delta$

Classes of points and sets

Open balls: $B(a,r) = \{x \in X : d(a,x) < r\}$

Closed balls: $\overline{B}(a,r) = \{x \in X : d(a,x) \leq r\}$

Interior, exterior and boundary points:



Interior point: There exists an $r > 0$ such that $B(a,r) \subseteq A$

Exterior point: There exists an $r > 0$ such that $B(a,r) \subseteq A^c$

Boundary point: All balls $B(a,r)$ contain points from both A and A^c

Open set: No boundary points belong to A

Closed set: All — " —

Affirmative descriptions:

A is open $\Leftrightarrow$ all points in A are interior

$\Leftrightarrow$ A^c is closed

A is closed $\Leftrightarrow$ A^c is open $\Leftrightarrow$ if a sequence from A converges to a, then $a \in A$

<u>Compact sets</u>: A is compact if all sequences in A have subsequences converging to points in A.

Alternative descriptions:

1 A compact $\Leftrightarrow$ A has the open covering property (i.e. every covering of A by open sets has a finite subcovering).

2 (requires that X is complete) A is compact $\Leftrightarrow$ A is closed and

<u>Special cases</u>:

In $\mathbb{R}^n$: compact $\Leftrightarrow$ closed and bounded

In $C(X, \mathbb{R}^n)$: compact $\Leftrightarrow$ closed, bounded and equicontinuous (Arzela-Ascoli's Th)

<u>Connections to continuous functions</u>

f continuous $\Leftrightarrow$ $f^{-1}(O)$ is open for all open O $\Leftrightarrow$ $f^{-1}(F)$ is closed for all closed F

If f is continuous and K is compact, then $f(K)$ is compact.

More about continuity

f is continuous at $a \Leftrightarrow f(x_n) \to f(a)$ for all sequences x_n converging to a.

Uniform continuity: For every $\varepsilon > 0$ there is a $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ whenever $d_X(x, y) < \delta$ (NB: The same δ works for all pairs x, y)

Important result: If K is compact, all continuous functions are uniformly continuous.

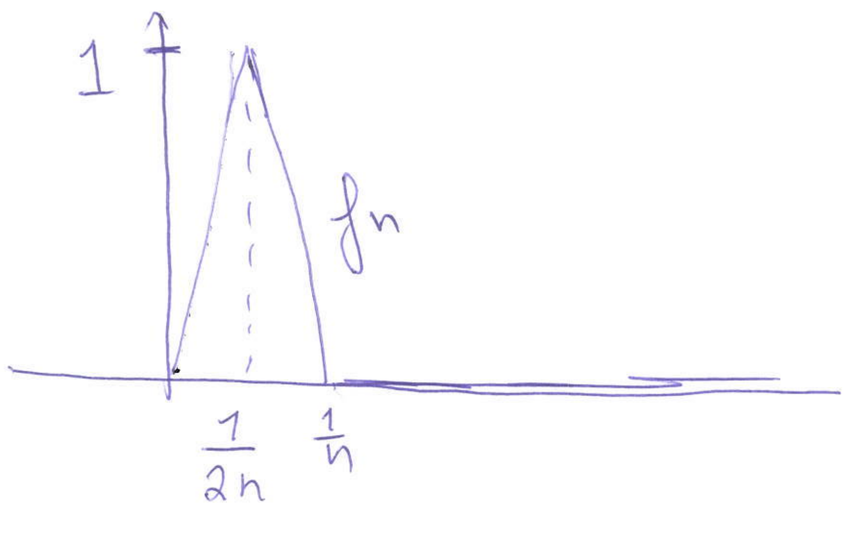
Equicontinuity: ~~...~~ A family $\mathcal{F}$ of functions from X to Y is equicontinuous if for all $\varepsilon > 0$ there is a $\delta > 0$ such that if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$ for all ~~...~~ $f \in \mathcal{F}$.

Pointwise and uniform convergence

Pointwise convergence: $f_n(x) \to f(x)$ for all x (i.e. given $x \in X$ and $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $d_Y(f_n(x), f(x))$ for all $n \geq N$)

Uniform convergence: For each $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that if $n \geq N$, then $d_Y(f_n(x), f(x)) < \varepsilon$ for all x (NB: The same N holds for all x)

Example of pointwise but not uniform convergence



Standard method for checking uniform conv.

(i) Find $\lim_{n \to \infty} f_n(x)$ using e.g. L'Hopital's rule. If a limit $f(x)$ exists for all x, then $\{f_n\}$ converges pointwise to f

(ii) To check if the convergence is uniform find
$d_n = \sup\{d(f_n(x), f(x)) : x \in X\}$
(usually a maximum problem that can be solved by differentiation). If $d_n \to 0$, the convergence is uniform, if $d_n \not\to 0$, then the convergence is not uniform.

Completeness

Key concept: $\{x_n\}$ is a Cauchy sequence if for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that if $n, m \geq N$, then $d(x_n, x_m) < \varepsilon$.

Completeness: X is complete if all Cauchy sequences converges

---

Examples: $\mathbb{R}, \mathbb{R}^n, C(X, Y)$ (for complete Y), $L^1(\mu)$ and $\cancel{L}^2(\mu)$ are complete.

$\mathbb{Q}, \mathbb{Q}^n$ are not complete

Results: All compact spaces are complete (but not the converse)

A closed subset of a complete space is complete.