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MAT2400

More about completeness

Banach's Fixed Point Theorem: Assume that (X,d) is a complete metric space and that $f: X \to X$ is a contraction. Then f has a unique fixed point a, and for any initial point x_0, the sequence $x_0, x_1 = f(x_0), f^{(2)}(x_0), \dots$ converges to a.

Idea of proof: Show that the sequence is a Cauchy sequence, show that the limit is a fixed point, and check that there isn't more than one fixed point.

Application: Existence of unique, global solutions to systems of differential equations

How does one prove that a space is complete?

Usually by checking the definition, perhaps by exploiting that another space is known to be complete.

Alternative 1: If (X,d) is complete and $A \subseteq X$ is closed, then the subspace (A, d_A) is also complete.

Alternative 2: A normed space is complete if and only if all absolutely convergent sequences converge

More about compactness

Some important properties of compact sets/spaces:

1 A compact set is closed and bounded (the converse holds in $\mathbb{R}^n$, but not in general)

2 A closed subset of a compact set is compact.

3 A compact space is complete (but $\mathbb{R}^n$ is complete but not compact)

4 Extreme Value Theorem: Assume that $K \subseteq X$ is compact and that $f: K \to \mathbb{R}$ is continuous. Then f is bounded and has maximal and minimal points

The proof is typical: Let

$$M = \sup\{f(x) : x \in K\}$$

and pick a sequence $\{x_n\}$ from K such that $f(x_n) \to M$. Since K is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to a point $a \in K$. Thus $f(a) = \lim_{k \to \infty} f(x_{n_k}) = M$, which shows that a is a maximum point

Dense and nowhere dense sets

Definisjon: $A \subseteq X$ is dense if for every $x \in X$ there is a sequence $\{a_n\}$ from A converging to x.

Equivalently: A is dense if every nonempty, open subset of X contains elements from A.

Examples: $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$

The polynomials are dense in $C([a,b])$ (Weierstrass' Theorem)

If $G \subseteq X$ is open, we say that A is dense in A if all balls $B(a;r) \subset G$ contains elements from A.

Definisjon: A is nowhere dense if it isn't dense in any open set. A set is meager if it is a countable union of nowhere dense sets.

Baire's Category Theorem:

In a complete space, a meager set does not contain any balls; i.e. M^c is dense.

Function spaces

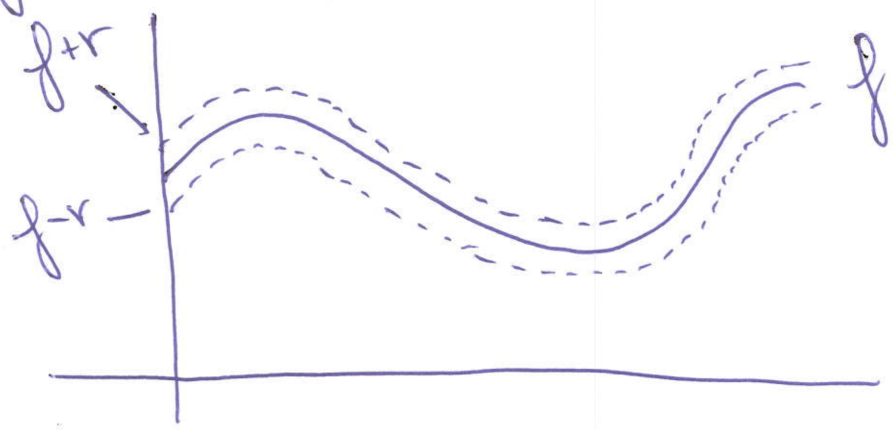
Assume that X, Y are metric spaces, X compact.

Then

$$\rho(f,g) = \sup\{d_Y(f(x),g(x)) : x \in X\}$$

is a metric on the space $C(X,Y)$ of continuous functions from X to Y.

The ball $B(f,r)$ around a function f looks like a sausage:



Convergence in the ρ-metric is the same as uniform convergence.

If Y is complete, so is $C(X,Y)$

$$\left.\begin{array}{l} L^1(X) = \{f : X \to \overline{\mathbb{R}} : \int |f| d\mu < \infty\}, \\ \text{norm } \|f\|_1 = \int |f| d\mu. \\ L^2(X) = \{f : X \to \overline{\mathbb{R}} : \int |f|^2 d\mu < \infty\} \\ \text{norm } \|f\|_2 = \left(\int |f|^2 d\mu\right)^{1/2} \end{array}\right\} \text{complete}$$

More about normed spaces

A sequence $\{\vec{e}_n\}$ is a basis if every $\vec{u} \in X$ can be written as a linear combination

$$\vec{u} = \sum_{n=1}^{\infty} \alpha_n \vec{e}_n$$

in a unique way.

If $\{\vec{e}_n\}$ is an orthonormal basis in an inner product space, then

$$\vec{u} = \sum_{n=1}^{\infty} \underbrace{\langle \vec{u}, \vec{e}_n \rangle}_{\text{Fourier coefficients}} \vec{e}_n$$

Linear operators: If X, Y are normed spaces, a function $A: X \to Y$ is a linear operator if

(i) $A(\alpha \vec{x}) = \alpha A(\vec{x})$ for all $\alpha \in K$, $\vec{x} \in X$

(ii) $A(\vec{x} + \vec{y}) = A(\vec{x}) + A(\vec{y})$ for all $\vec{x}, \vec{y} \in X$

A is bounded if there is an $M \in \mathbb{R}$ such that $\|A(\vec{x})\|_Y \leq M \|\vec{x}\|_X$ for all $\vec{x} \in X$.

Theorem: The following are equivalent.

(i) A is bounded

(ii) A is continuous at $\vec{0}$

(iii) A is uniformly continuous

Measure spaces

σ-algebra on X: Collection $\mathcal{A}$ of subsets of X such that:

(i) $\emptyset \in \mathcal{A}$

(ii) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$

(iii) If $A_1, A_2, \ldots \in \mathcal{A}$, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$

Measure μ on $\mathcal{A}$:

(i) $\mu(\emptyset) = 0$

(ii) If $\{A_n\}$ is a disjoint sequence of sets in $\mathcal{A}$, then $\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$.

Continuity of measures:

(i) If $\{A_n\}$ is an increasing sequence of sets in $\mathcal{A}$, then $\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \lim_{n \to \infty} \mu(A_n)$

(ii) If $\{A_n\}$ is a decreasing sequence of sets in $\mathcal{A}$ and $\mu(A_1) < \infty$, then $\mu\left(\bigcap_{n \in \mathbb{N}} A_n\right) = \lim_{n \to \infty} \mu(A_n)$

Complete measures: μ is complete if whenever $\mu(A) = 0$ and $N \subseteq A$, then $N \in \mathcal{A}$ and $\mu(N) = 0$.
All measures can be extended to complete measures

<u>Measurable functions</u>:

$$f^{-1}([-\infty, r)) = \{x \in X : f(x) < r\} \in \mathcal{A}.$$

Implies that $f^{-1}(A) \in \mathcal{A}$ for all open and closed A.

Sums, products, limits etc of measurable functions are measurable.

<u>Simple function</u>: Measurable functions that take only finitely many of values:

$$f = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}$$

<u>Integration</u> of simple functions

$$\int f \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$$

(this the definition when f is on standard form and a proposition otherwise. The distinction is important in the development of the theory, but not later)