MAT 2400

30/6 - 2014

Synopsis 3
Integration

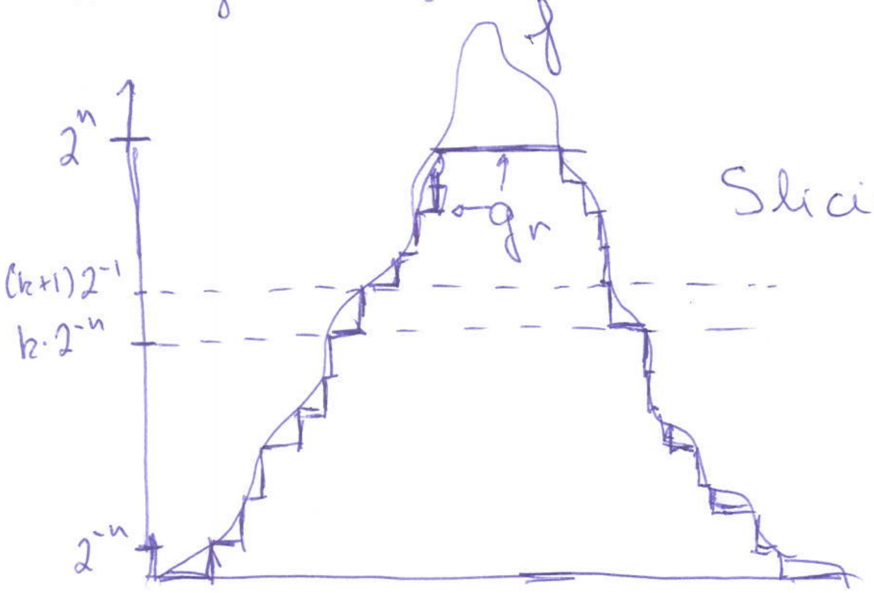
Integration of nonnegative step functions:

$f = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}$, $\int f \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$ holds even when f is not written on standard form.

Integration of nonnegative, measurable functions:

$\int f \, d\mu = \sup\{\int g \, d\mu : g \leq f, \ g$ a nonnegative step function$\}$

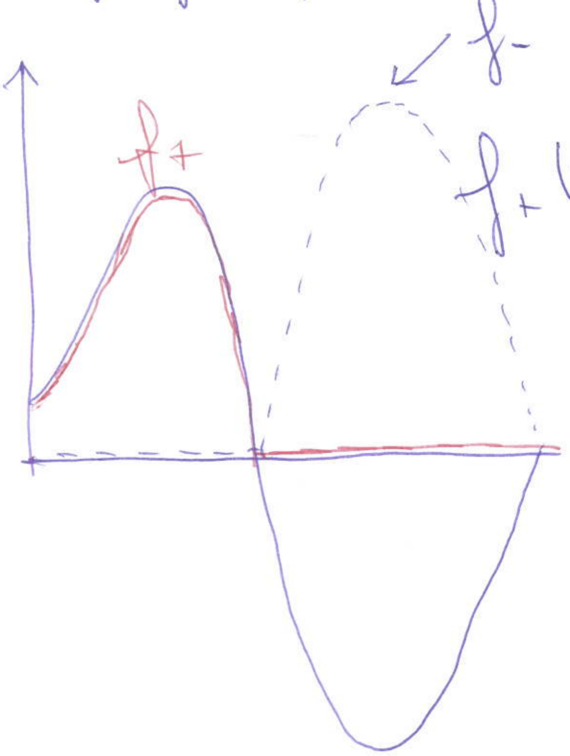
Alternatively: $\int f \, d\mu = \lim_{n \to \infty} \int g_n \, d\mu$ where $\{g_n\}$ is any increasing sequence of nonnegative step functions converging to f. Such a sequence always exists.



Slicing up the y-axis

Integration of measurable functions

$f: X \to \overline{\mathbb{R}}$: Split in a positive and negative part $f = f_+ - f_-$



$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0 \\ 0 & \text{otherwise} \end{cases}$$

<u>f is integrable if</u>: $\int f_+ \, d\mu < \infty$ and $\int f_- \, d\mu < \infty$.

Alternatively $\int |f| \, d\mu < \infty$

<u>Definition</u>: If f is integrable, then

$$\int f \, d\mu = \int f_+ \, d\mu - \int f_- \, d\mu$$

<u>Basic properties of the integral</u>

a) $\int c f \, d\mu = c \int f \, d\mu$, $c \in \mathbb{R}$

b) $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$

c) If $f \leq g$, then $\int f \, d\mu \leq \int g \, d\mu$

Limit theorems

Monotone Convergence Theorem. Assume that $\{f_n\}$ is an increasing sequence of nonnegative measurable functions converging almost everywhere to a measurable function f. Then

$$\int f\,d\mu = \lim_{n\to\infty}\int f_n\,d\mu$$

Series version: Assume that $\{u_n\}$ is a sequence of nonnegative, measurable functions. Then

$$\int \sum_{n=1}^{\infty} u_n\,d\mu = \sum_{n=1}^{\infty}\int u_n\,d\mu$$

Lebesgue's Dominated Convergence Theorem: Assume that $\{f_n\}$ is a sequence of measurable functions converging almost everywhere to a measurable function f. If there is an integrable function g such that $|f_n| \le g$ for all n, then

$$\int f\,d\mu = \lim_{n\to\infty}\int f_n\,d\mu$$

Remark: Note that if $f_n \to f$ pointwise, then f is automatically measurable.

Fatou's Lemma: If $\{f_n\}$ is a sequence of non-negative, measurable functions, then

$$\liminf \int f_n\,d\mu \ge \int \liminf f_n\,d\mu$$

One may also regard the completeness of L^1 and L^2 as limit theorems:

If $\{f_n\}$ is a Cauchy sequence in L^1, then $\{f_n\}$ converges to a function f in L^1-norm, i.e. $\int |f_n - f| d\mu \to 0$. Since

$$\left|\int f_n d\mu - \int f d\mu\right| = \left|\int (f_n - f) d\mu\right| \leq \int |f_n - f| d\mu \to 0,$$

this means that

$$\lim \int f_n d\mu = \int f d\mu$$

The same holds in L^2: If $\{f_n\}$ is a Cauchy sequence, it converges to an $f \in L^2$, and

$$\lim_{n\to\infty} \int f_n^2 d\mu = \int f d\mu.$$

Construction of measures

Starting point: A nonempty collection $\mathcal{R}$ of subsets of X and a function $\rho: \mathcal{R} \to \mathbb{R}_+$ such that

(i) $\emptyset \in \mathcal{R}$ and $\rho(\emptyset) = 0$

(ii) There is a collection $\{R_n\}_{n\in\mathbb{N}}$ of sets in $\mathcal{R}$ such that $\bigcup_{n\in\mathbb{N}} R_n = X$.

We define the outer measure by

$$\mu^*(A) = \inf\left\{ \sum_{n=1}^{\infty} \rho(R_n) : \{R_n\} \text{ is an } \mathcal{R}\text{-covering of } A \right\}$$

Properties of the outer measure:

(i) $\mu^*(\emptyset) = 0$

(ii) $\mu^*(R) \leq \rho(R)$ for all $R \in \mathcal{R}$

(iii) $\mu^*(C) \leq \mu^*(D)$ if $C \subseteq D$

(iv) $\mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$ for all sequences $\{A_n\}$ of sets.

Measurable sets: E is measurable if for all $A \subseteq X$:

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Theorem: The measurable sets form a σ-algebra $\mathcal{A}$, and the restriction $\mu = \mu^*\restriction_{\mathcal{A}}$ of μ^* to $\mathcal{A}$ is a measure.

Problem: $\mathcal{A}$ and μ need not be extensions of $\mathcal{R}$ and ρ.

Definition: We call μ a measure extension of ρ if $\mathcal{R} \subseteq \mathcal{A}$ and $\mu(R) = \rho(R)$ for all $R \in \mathcal{R}$.

We can only hope to find a measure extension if ρ is a premeasure, i.e. if

(i) $\rho(\emptyset) = 0$ (already assumed)

(ii) Whenever $\{R_n\}$ is a disjoint sequence from $\mathcal{R}$ whose union happens to be in $\mathcal{R}$, then

$$\rho\left(\bigcup_{n=1}^{\infty} R_n\right) = \sum_{n=1}^{\infty} \rho(R_n)$$

We also need conditions on $\mathcal{R}$: We say that $\mathcal{R}$ is a <u>semi-algebra</u> if

(i) If $R, S \in \mathcal{R}$, then $R \cap S \in \mathcal{R}$

(ii) If $R \in \mathcal{R}$, then R^c is a disjoint, finite union $R^c = S_1 \cup S_2 \cup \ldots \cup S_n$ of sets in $\mathcal{R}$.

<u>Examples</u> of semi-algebras:

1) The collection of all intervals of the forms $(a,b]$, (a,∞), $(-\infty,b]$.

2) The collection of measurable rectangles $A \times B$ when $A \in \mathcal{A}$, $B \in \mathcal{B}$.

<u>Caratheodory's Extension Theorem</u>: A premeasure ρ defined on a semi-algebra $\mathcal{R}$ has a measure extension obtained by the outer measure construction.

<u>Applications</u>:

1) Use the intervals in Example 1) above with $\rho(I) = $ length of I. The generated measure is the one-dimensional Lebesgue measure

2) Use the measurable rectangles in Example 2) with $\rho(A \times B) = \mu(A)\nu(B)$. The generated measure is the product measure $\mu \times \nu$.

Tonelli-Fubini's Theorem: Assume that $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite and that $f: X \times Y \to \overline{\mathbb{R}}$ is $\mathcal{A} \otimes \mathcal{B}$-measurable.

(i) (Tonelli) If $f \geq 0$, then

$$\int f \, d\mu \times \nu = \int \left[\int f(x,y) \, d\mu(x) \right] d\nu(y)$$

$$(*) \qquad = \int \left[\int f(x,y) \, d\nu(y) \right] d\mu(x)$$

(ii) (Fubini) If f is integrable, then

$$\int f \, d\mu \times \nu = \int \left[\int f(x,y) \, d\mu(x) \right] d\nu(y)$$

$$(**) \qquad = \int \left[\int f(x,y) \, d\nu(y) \right] d\mu(x)$$

How to use Fubini's Theorem: Usually, one wants to apply the theorem to compute the integral $\int f \, d\mu \times \nu$. In most cases, it is hard to compute this integral directly, but much easier to compute the iterated integrals $\int [\int f(x,y) \, d\mu(x)] d\nu(y)$ or $\int [\int f(x,y) \, d\nu(y)] d\mu(x)$. The problem is that Fubini's Theorem allows you to interchange these problems provided that f is integrable, something you probably don't know. To solve this problem you first apply Tonelli's Theorem to $|f|$ – this you can do without any knowledge of integrability. If your calculations show that either $\int[\int|f(x,y)|d\mu(x)]d\nu(y)$ or $\int[\int|f(x,y)|d\nu(y)]d\mu(x)$ (you can choose the one that is easiest to compute) is finite, Tonelli's Theorem tells you that f is integrable (since $\int|f|d\mu\times\nu<\infty$) and hence you can use Fubini's Theorem the way you intended. It is quite common to use Tonelli's and Fubini's Theorem in tandem in this way.