UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT2400 — Real analysis

Day of examination: Thursday, June 2, 2015

Examination hours: 14:30 – 18:30

This problem set consists of 5 pages.

Appendices: None.

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Let X be the space of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with the supremum metric

$$d_\infty(f, g) = \sup_{x\in\mathbb{R}} |f(x) - g(x)| .$$

1a

Show that d_∞ defines a metric on X.

Possible answer: This is easy.

1b

Set $f_r(x) = f(x + r)$ for $r \in \mathbb{R}$. Show that if $f \in X$ and f is uniformly continuous, then $\lim_{r\to 0} d_\infty(f_r, f) = 0$.

Possible answer: Given $\varepsilon > 0$, we have to find a δ such that $|r| < \delta$ implies that $d_\infty(f, f_r) \le \varepsilon$. Since f is uniformly continuous, we can find a δ such that $|f(x) - f(x + r)| \le \varepsilon$ for all $|r| < \delta$ and for all $x \in \mathbb{R}$. Then

$$|r| \le \delta \Rightarrow |f(x + r) - f(x)| \le \varepsilon \text{ for all } x.$$

Then this inequality holds also for the supremum.

1c

For $x \in \mathbb{R}$, let $g(x) = \cos(x^2\pi)$. Show that g is not uniformly continuous. (Hint: As x grows, g will oscillate more and more rapidly.)

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Possible answer: For $n \in \mathbb{N}$ we have that

$$\left| g\left(\sqrt{n} + \frac{1}{\sqrt{n+1}+\sqrt{n}}\right) - g(\sqrt{n}) \right| = 2.$$

Hence, for any $\delta > 0$, we can find n such that

$$p := \frac{1}{\sqrt{n+1}+\sqrt{n}} \leq \delta,$$

and then

$$\sup_{x\in\mathbb{R}} \sup_{|r|\leq\delta} |g(x) - g(x+r)| \geq \left|g(\sqrt{n}+p) - g(\sqrt{n})\right| = 2.$$

Thus g is not uniformly continuous.

1d

Is it true that $\lim_{r\to 0} d_\infty(f_r, f) = 0$ for all $f \in X$?

Possible answer: No, not for the function in the previous question.

Problem 2

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function, and set

$$\operatorname{sign}(u) = \begin{cases} 1 & u > 0, \\ 0 & u = 0, \\ -1 & u < 0. \end{cases}$$

Show that the composite function $s(x) = \operatorname{sign}(f(x))$ is measurable.

Possible answer: The function s is measurable if $s^{-1}\{[-\infty, r)\}$ is measurable for all r. Now

$$s^{-1}\{[-\infty, r)\} = \begin{cases} \mathbb{R}^d & r > 1, \\ \{x \mid f(x) < 0\} & -1 < r \leq 1, \\ \emptyset & r \leq -1. \end{cases}$$

The upper and lower sets are measurable, and the middle set is measurable since f is measurable.

Problem 3

Let X be a vector space with norm $\|\cdot\|$, and let $V \subseteq X$ be a linear subspace (i.e., V is also a vector space with norm $\|\cdot\|$), such that $\operatorname{int}(V) \neq \emptyset$. Show that $V = X$. ($\operatorname{int}(V)$ is the set of interior points in V)

(Continued on page 3.)

Possible answer: Let $u \in \text{int}(V)$, then there is an $r > 0$ such that $B_r(u) \subseteq V$. Let $z \in B_r(u)$, since V is a vector space, $z - u \in V$, but $z - u \in B_r(0)$. Hence $B_r(0) \subseteq V$. Let $x \in X$, then for $\rho \leq r$, $y \in B_r(0)$, where

$$y = \frac{\rho}{\|x\|} x.$$

Since V is a vector space $x = \frac{\|x\|}{\rho} y \in V$. Hence $V = X$.

Problem 4

Let $C[0,1]$ denote the space of continuous functions from the interval $[0,1]$ with values in $\mathbb{R}$. Let Lu be defined by

$$(Lu)(t) = \int_0^1 \frac{1}{1+t+s} f(u(s))\, ds,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function.

4a

Show that L maps $C[0,1]$ into $C[0,1]$.

Possible answer: We have that

$$\begin{aligned} |(Lu)(\tau) - (Lu)(t)| &\leq \int_0^1 \left| \frac{1}{1+\tau+s} - \frac{1}{1+t+s} \right| M\, ds \\ &\leq M\,|\tau - t| \int_0^1 \left| \frac{1}{(1+s)^2} \right| ds \\ &= \frac{1}{2} M\,|\tau - t|\,, \end{aligned}$$

where M is a bound on $|f|$. Thus Lu is continuous.

4b

Assume now that

$$|f(u) - f(v)| < \frac{1}{\ln(2)}\,|u - v| \quad \text{for all } u \text{ and } v.$$

Show that the equation $Lu = u$ has a unique solution in $C[0,1]$.

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Possible answer: We show that L is a contraction in the supremum norm.

$$\begin{aligned}
|(Lu)(t) - (Lv)(t)| &< \int_0^1 \frac{1}{1+t+s}\frac{1}{\ln(2)}\,|u(s)-v(s)|\;ds \\
&< \frac{1}{\ln(2)} \sup_{t\in[0,1]} |u(t)-v(t)| \int_0^1 \frac{1}{1+t+s}\,ds \\
&= \frac{1}{\ln(2)} \sup_{t\in[0,1]} |u(t)-v(t)| \ln\left(\frac{2+t}{1+t}\right) \\
&\le \sup_{t\in[0,1]} |u(t)-v(t)|\,,
\end{aligned}$$

since $t \geq 0$. Therefore L is a contraction, and $Lu = u$ has a unique solution.

Problem 5

Let the function $f : [-\pi, \pi] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{\sin(x)}{x}, & x \neq 0, \\ 1 & x = 0, \end{cases}$$

and for x outside $[-\pi, \pi]$ f is the periodic extension.

5a

Show that

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi}\int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin(x)}{x}\,dx.$$

(Hint: write $\sin(x) = (e^{ix} - e^{-ix})/(2i)$ and use the change of variables $z = (n+1)x$ and $z = (n-1)x$.).

Possible answer: The periodic extension is continuous, hence the Fourier series converges pointwise. Therefore we have the equality.
To compute the coefficients,

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(x)}{x} e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{ix(1-n)} - e^{-ix(1+n)}}{2ix}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2ix} e^{-ix(n-1)}\,dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2ix} e^{-ix(n+1)}\,dx.
\end{aligned}$$

Change variables $z = x(n-1)$ in the first integral, and $z = x(n+1)$ in the

(Continued on page 5.)

second to get

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\Big(\int_{-(n-1)\pi}^{(n-1)\pi} \frac{1}{2iz}e^{-iz}\,dz - \int_{-(n+1)\pi}^{(n+1)\pi} \frac{1}{2iz}e^{-iz}\,dz\Big)\\
&= \frac{1}{2\pi}\Big(\int_{-(n+1)\pi}^{-(n-1)\pi} \frac{1}{2iz}e^{-iz}\,dz - \int_{(n-1)\pi}^{(n+1)\pi} \frac{1}{2iz}e^{-iz}\,dz\Big) \text{ } (z \mapsto -z \text{ in first integral})\\
&= \frac{1}{2\pi}\Big(\int_{(n-1)\pi}^{(n+1)\pi} \frac{1}{2iz}e^{iz}\,dz - \int_{(n-1)\pi}^{(n+1)\pi} \frac{1}{2iz}e^{-iz}\,dz\Big)\\
&= \frac{1}{2\pi}\int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin(z)}{z}\,dz.
\end{aligned}
$$

5b

Use this to compute the integral

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx.$$

Possible answer: We know that the series converges for $x = 0$, hence

$$
\begin{aligned}
1 &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-(n-1)\pi}^{(n+1)\pi} \frac{\sin(z)}{z}\,dz\\
&= \frac{1}{2\pi}\,2\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx.
\end{aligned}
$$

Therefore the integral equals π.

THE END