UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT2400 — Real analysis

Day of examination: Thursday, August 13, 2015

Examination hours: 14:30 – 18:30

This problem set consists of 6 pages.

Appendices: None.

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

1a

Let $\omega_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$\omega_n(x) = \begin{cases} n(1 - n\,|x|) & |x| \le 1/n, \\ 0 & \text{otherwise,} \end{cases}$$

for $n = 1, 2, 3, \ldots$. Find $\lim_{n\to\infty} \omega_n(x)$. Does $\{\omega_n\}$ converge in the $L^1(\mathbb{R})$ norm ($\|f\|_1 = \int |f|\, d\mu$)?

Possible answer:

$$\lim_{n\to\infty} \omega_n(x) = \begin{cases} 0 & x \ne 0, \\ \infty & x = 0. \end{cases}$$

We have that $\int |\omega_n|\, d\mu = 1$, therefore ω_n does not converge in L^1.

1b

Let $g : \mathbb{R} \to \mathbb{R}$ be an integrable function such that $\sup_{x\in\mathbb{R}} |g(x)| \le K$, and $\int |g|\, d\mu = L < \infty$. Define

$$g_n(x) = \int g(y)\omega_n(x - y)\, d\mu(y).$$

Show that $\sup_{x\in\mathbb{R}} |g_n(x)| \le K$, and that g_n is continuous for each (finite) n.

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Possible answer: We have that

$$|g_n(x)| \leq \int |g(y)|\, \omega_n(x-y)\, d\mu(y)$$
$$\leq \sup_{z\in\mathbb{R}^d} |g(z)| \int \omega_n(x-y)\, d\mu(y) = K,$$

for all x. Hence $\sup_{x\in\mathbb{R}} |g_n(x)| \leq 1$. To show continuity, we observe that ω_n is (Lipschitz) continuous,

$$|\omega_n(a) - \omega_n(b)| \leq \frac{1}{n^2} |a-b|.$$

Then

$$|g_n(x) - g_n(z)| \leq \int |g(y)|\, |\omega_n(x-y) - \omega_n(z-y)|\, d\mu(y)$$
$$\leq \frac{|x-z|}{n^2} \int |g|\, d\mu = \frac{L}{n^2} |x-z|.$$

1c

Assume now that g is continuous. Show that $g_n(x) \to g(x)$ for each x (pointwise convergence) and that $g_n \to g$ in $L^1(\mathbb{R})$ as $n \to \infty$. (This means that $\|g_n - g\|_1 = \int |g_n - g|\, d\mu \to 0$ as $n \to \infty$.)

Possible answer: We have that

$$|g(x) - g_n(x)| \leq \int \omega_n(x-y)\, |g(x) - g(y)|\, d\mu \leq \sup_{y\in B(x,1/n)} |g(x) - g(y)|.$$

Therefore

$$\inf_{y\in B(x,1/n)} g(y) \leq g_n(x) \leq \sup_{y\in B(x,1/n)} g(y).$$

Since g is continuous both of these will converge to $g(x)$ as $n \to \infty$.
Set $f_n = |g - g_n|$, then $f_n \to 0$, and

$$\int f_n\, d\mu \leq \int |g|\, d\mu + \int |g_n|\, d\mu \leq 2 \int g\, d\mu$$

since

$$\int |g_n|\, d\mu \leq \int \int \omega_n(x-y)\, |g(y)|\, d\mu(y)\, d\mu(x) = \int |g(y)|\, d\mu(y),$$

because $\int \omega_n(x-y)\, d\mu(x) = 1$. Therefore we can use the dominated convergence theorem to conclude that

$$\int |g - g_n|\, d\mu \to 0.$$

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Problem 2

For $n \in \mathbb{N}$, define the function

$$\varphi_n(x) = \begin{cases} (1 - e^n |x|) & |x| \le e^{-n}, \\ 0 & \text{otherwise.} \end{cases}$$

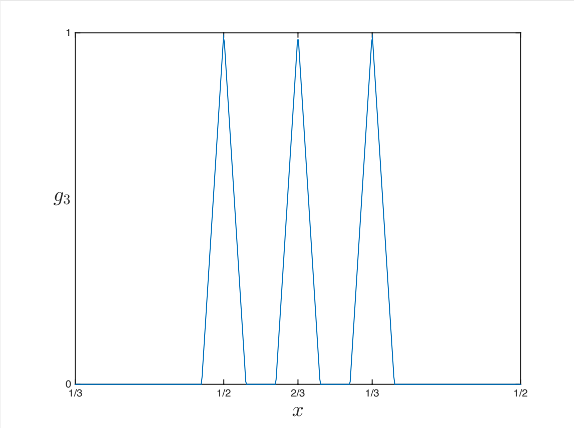
Let $R_n = \{r_{n,1}, r_{n,2}, \ldots, r_{n,K}\}$ be the set of rational numbers in $(0, 1)$ with denominator less than or equal to n. Note that K depends on n. Define the functions

$$g_n(x) = \sum_{k=1}^{K} \varphi_n(x - r_{n,k}).$$

2a

Sketch the graph of g_3.

Possible answer: The rational numbers in $(0, 1)$ with denominator less than 3 are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{2}{3}$. Therefore $R_3 = \left\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right\}$. Furthermore $e^{-3} \approx 0.05 < \frac{1}{2}(\frac{1}{2} - \frac{1}{3}) = \frac{1}{12}$. The graph looks like this



2b

Show that if $r \in \mathbb{Q} \cap (0, 1)$ then $g_n(r) \to 1$.

Possible answer: If $(0, 1) \ni r = p/q$ with $p \le q$, then $r \in R_q \subseteq R_n$ for $n \ge q$. Also, $\varphi_N(r_{q,k} - r_{q,j}) = 0$ for $j \ne k$ since

$$|r_{q,j} - r_{q,k}| \ge \frac{1}{q-1} - \frac{1}{q} > e^{-q}.$$

Therefore $g_q(r_{q,k}) = 1$, and by the same argument $g_n(r) = 1$ for all $n \ge q$. If $r \in \mathbb{Q} \cap (0, 1)$ then $r \in R_n$ for all $n \ge N$ for some N. Thus $\lim_{n\to\infty} g_n(r) = 1$.hag

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2c

Show that

$$\int_0^1 |g_n(x)|\, dx \to 0,$$

as $n \to \infty$.

Possible answer: Regarding the integral

$$\int g_n\, dx \leq \sum_{k=1}^{K} \int \varphi_n(x - r_{n,k})\, dx \leq K2e^{-n}.$$

Now we must estimate K, i.e., how many rational numbers there are in $(0,1)$ with denominator less than n. We can overestimate this by $1+2+\cdots+n = n(n+1)/2$. Then

$$\int_0^1 |g_n(x)|\, dx \leq n(n+1)e^{-n} \to 0, \text{ as } n \to \infty.$$

Problem 3

Let $\mathbb{R}^*$ denote the extended real numbers, i.e., $\mathbb{R}^* = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$, and define $f : \mathbb{R}^* \to [-1, 1]$ by

$$f(x) = \begin{cases} -1 & x = -\infty, \\ \frac{x}{1+|x|} & x \in \mathbb{R}, \\ 1 & x = \infty. \end{cases}$$

3a

Show that f is $1-1$ and onto.

Possible answer: For $x \in \mathbb{R}$, f is strictly increasing and $f(\mathbb{R}) = (-1, 1)$, furthermore $f^{-1}(\pm 1) = \pm\infty$. Therefore f is a bijection.

3b

For x and y in $\mathbb{R}^*$, define $d(x, y) = |f(x) - f(y)|$. Show that d is a distance on $\mathbb{R}^*$.

Possible answer: Clearly $d \geq 0$, $d(x, y) = d(y, x)$ and $d(x, y) = 0$ if and only if $x = y$ since f is increasing. We must show the triangle inequality. By symmetry, we can assume that $x < y$ (x and y in $\mathbb{R}^*$). If $x \leq z \leq y$ then

$$d(x, y) = f(y) - f(x) = f(y) - f(z) + f(z) - f(x) = d(x, z) + d(z, y).$$

(Continued on page 5.)

Then assume that $x \leq y \leq z$, then

$$\begin{aligned} d(x,y) &= f(y) - f(x) = f(z) - f(x) + f(y) - f(z) \\ &= d(x,z) - d(y,z) \leq d(x,z) + d(z,y), \end{aligned}$$

since f is increasing. The case where $z \leq x \leq y$ is similar.

Problem 4

Let X be a compact metric space and let $\{f_n\}_{n=0}^{\infty}$ be a sequence of continuous functions $f_n : X \to \mathbb{R}$ such that

$$0 \leq f_0 \leq f_1 \leq f_2 \leq \cdots \leq f_n \leq f_{n+1} \leq \cdots .$$

Assume that $f(x) = \lim_{n\to\infty} f_n(x)$ is continuous and $f(x) < \infty$ for all $x \in X$.

4a

Show that f_n converges uniformly to f.

Possible answer: Since both f_n and f are continuous, so is $|f - f_n|$. Therefore, given an $\varepsilon > 0$, the set

$$U_n = \{x \in X \mid |f_n(x) - f(x)| < \varepsilon\}$$

is open. Since f_n converges to f for all $x \in X$, $X = \bigcup_{n=0}^{\infty} U_n$, but since X is compact a finite union will cover X, i.e., $X = \bigcup_{n=0}^{N(\varepsilon)} U_n$. Since $f_n \leq f_{n+1} \leq f$, $U_n \subseteq U_{n+1}$. Hence $X = \bigcup_{n=0}^{N(\varepsilon)} U_n = U_{N(\varepsilon)}$. Therefore, for $n \geq N(\varepsilon)$, $|f(x) - f_n(x)| \leq \varepsilon$ for all $x \in X$. This is uniform convergence.

4b

Define a sequence of polynomials on $[0,1]$ as follows: $p_0(x) = 0$,

$$p_{n+1}(x) = p_n(x) + \frac{1}{2}\left(x - p_n^2(x)\right), \qquad n \geq 0.$$

Show that $p_n(x)$ converges uniformly to $f(x) = \sqrt{x}$.

Possible answer: To use the previous question, we must show that $p_n \leq p_{n+1}$. Assume inductively that $0 \leq p_n(x) \leq \sqrt{x} \leq 1$. This holds for $n = 0$. Assume that it holds for n, then $x \geq p_n^2$ and we get

$$p_{n+1}(x) = p_n(x) + \frac{1}{2}\left(x - p_n(x)^2\right) \geq p_n(x) \geq 0,$$

and

$$\begin{aligned} p_{n+1}(x) &= p_n(x) + \frac{1}{2}\left(\sqrt{x} + p_n(x)\right)\left(\sqrt{x} - p_n(x)\right) \\ &\leq p_n(x) + \frac{1}{2}(1+1)\left(\sqrt{x} - p_n(x)\right) = \sqrt{x} \leq 1. \end{aligned}$$

(Continued on page 6.)

Set $f(x) = \lim_n p_n(x)$ (which exists since $p_n \leq p_{n+1} \leq 1$) then $f = f - \frac{1}{2}(x - f^2)$ which means that $f(x) = \sqrt{x}$. This is a continuous function on $[0, 1]$, and then the previous question implies that the convergence is uniform.

THE END