UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT2400 — Real analysis.

Day of examination: Thursday, August 14th, 2014.

Examination hours: 14.30 – 18.30

This problem set consists of 2 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All items (1a, 1b, 1c, 2a, etc.) count 10 points.

Problem 1

Define a function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d(\mathbf{x}, \mathbf{y}) = \max\{|y_1 - x_1|, |y_2 - x_2|\}$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$.

a) Show that d is a metric on $\mathbb{R}^2$.

b) Describe what a ball $\mathrm{B}(\mathbf{a}; r)$ in the d-metric looks like.

c) Show that $(\mathbb{R}^2, d)$ is complete.

Problem 2

In this problem, $\mathcal{A}$ is a σ-algebra on a nonempty set X, and $\mathcal{M}^+$ is the set of all nonnegative, measurable functions $f : X \to [0, \infty]$. Also, $I : \mathcal{M}^+ \to [0, \infty]$ is a function with the following properties:

(i) $I(\alpha f) = \alpha I(f)$ for all $\alpha \in [0, \infty)$ and all $f \in \mathcal{M}^+$.

(ii) $I(f + g) = I(f) + I(g)$ for all $f, g \in \mathcal{M}^+$.

(iii) If $\{f_n\}$ is an increasing sequence of functions in $\mathcal{M}^+$ that converges pointwise to f, then $\lim_{n\to\infty} I(f_n) = I(f)$.

a) Show that if $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0, \infty)$ and $f_1, f_2, \ldots, f_n \in \mathcal{M}^+$, then

$$I(\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n) = \alpha_1 I(f_1) + \alpha_2 I(f_2) + \cdots + \alpha_n I(f_n)$$

(Continued on page 2.)

Define $\mu : \mathcal{A} \to [0, \infty]$ by $\mu(A) = I(\mathbf{1}_A)$.

b) Show that $\mu(\emptyset) = 0$ and that $\mu(A_1 \cup A_2 \cup \ldots \cup A_n) = \mu(A_1) + \mu(A_2) + \ldots + \mu(A_n)$ for all *disjoint* sets $A_1, A_2, \ldots, A_n \in \mathcal{A}$.

c) Show that μ is a measure.

d) Show that $I(f) = \int f \, d\mu$ for all nonnegative, simple functions.

e) Show that $I(f) = \int f \, d\mu$ for all $f \in \mathcal{M}^+$.

Problem 3

In this problem we shall study the following result:

Theorem: *Assume that* (X, d) *is a compact metric space and that* $\{f_n\}$ *is a decreasing sequence of continuous functions* $f_n : X \to \mathbb{R}$ *converging pointwise to 0. Then* $\{f_n\}$ *converges uniformly.*

a) Use the theorem to prove the following claim:

   Claim: Assume that (X, d) is a compact metric space and that $\{g_n\}$ is an increasing sequence of continuous functions $g_n : X \to \mathbb{R}$ converging pointwise to a continuous function g. Then $\{g_n\}$ converges uniformly to g.

In the remainder of the problem, we shall prove the theorem by contradiction. We therefore assume that there exists a decreasing sequence $\{f_n\}$ of continuous functions converging pointwise, but not uniformly to 0.

b) Explain that there is an $\epsilon > 0$ such that for every $n \in \mathbb{N}$ there is a point $x_n \in X$ where $f_n(x_n) \geq \epsilon$.

c) Explain that $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to a point $x \in X$. Explain also that there is an $N \in \mathbb{N}$ such that $f_N(x) < \frac{\epsilon}{2}$.

d) Explain that there is a $\delta > 0$ such that if $n \geq N$, then $f_n(u) < \epsilon$ for all u such that $d(x, u) < \delta$.

e) Show that for all sufficiently large $k \in \mathbb{N}$, we have $f_{n_k}(x_{n_k}) < \epsilon$, and explain why this gives us the contradiction we are looking for.

GOOD LUCK!