MAT2400: Mandatory Assignment, Spring 2016

Deadline: *You must turn in your paper before 2.30 p.m., Thursday, March 10th, 2016, in the designated area on the 7th floor of NHA. Remember to use the official front page available on the 7th floor and at*

http://www.uio.no/studier/admin/obligatoriske-aktiviteter/mn-math-obligforside.pdf

If you due to illness or other circumstances need to extend the deadline, you must apply for an extension to studieinfo@math.uio.no *before the deadline. Remember that illness has to be documented by a medical doctor. See*

http://www.uio.no/studier/admin/obligatoriske-aktiviteter/mn-math-oblig.html

for more information about the rules for mandatory assignments.

Instructions: *The assignment is compulsory, and students who do not get their paper accepted, will not get access to the final exam. To get the assignment accepted, you need a score of at least 60%. In the evaluation, credit will be given for a clear and well-organized presentation. All questions (points 1a, 1b, 1c etc.) have equal weight. Students who do not get their original paper accepted, but who have made serious and documented attempts to solve the problems, will get* one *chance of turning in an improved version. In solving the problems you may collaborate with others and use tools of all kinds. However, the paper you turn in should be written by* you *(by hand or computer) and should reflect* your *understanding of the material. If we are not convinced that you understand your own paper, we may ask you to give an oral presentation.*

Problem 1.

a) Show that the series
$$\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$$
converges for $x > 0$ and diverges for $x = 0$.

b) Show that the series converges uniformly on $[a, \infty)$ for all $a > 0$.

c) Define $f : (0, \infty) \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$. Show that f is continuous.

d) Show that the series does not converge uniformly on $(0, \infty)$.

Problem 2. Let (X, d) be a metric space. For a subset A of X, let ∂A denote the set of all boundary points of A. The *closure* of A is the set $\bar{A} = A \cup \partial A$.

a) Show that $\bar{A}$ is closed.

b) A subset A of X is called *precompact* if $\bar{A}$ is compact. Show that A is precompact if and only if all sequences in A have convergent subsequences.

c) Show that a subset of $\mathbb{R}^m$ is precompact if and only if it is bounded.

Problem 3. A metric space (X, d) is said to be *disconnected* if there are two nonempty open set O_1, O_2 such that

$$X = O_1 \cup O_2 \quad \text{and} \quad O_1 \cap O_2 = \emptyset$$

A metric space that is *not* disconnected is said to be *connected*.

a) Let $X = (-1, 1) \setminus \{0\}$ and let d be the usual metric $d(x, y) = |x - y|$ on X. Show that (X, d) is disconnected.

b) Let $X = \mathbb{Q}$ and let again d be the usual metric $d(x, y) = |x - y|$ on X. Show that (X, d) is disconnected.

c) Assume that (X, d) is a connected metric space and that $f : X \to Y$ is continuous and surjective. Show that Y is connected.

A metric space (X, d) is called *path-connected* if for every pair x, y of points in X, there is a continuous function $r : [0, 1] \to X$ such that $r(0) = x$ and $r(1) = y$ (such a function is called a *path* from x to y).

d) Let d be the usual metric on $\mathbb{R}^n$:

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

Show that $(\mathbb{R}^n, d)$ is path-connected.

e) Show that every path-connected metric space is connected. (*Hint:* Argue contrapositively: Assume that (X, d) is not connected. Let O_1, O_2 be two nonempty, open sets such that $X = O_1 \cup O_2$ and $O_1 \cap O_2 = \emptyset$, and pick points $x \in O_1$, $y \in O_2$. Show that there doesn't exist a path from x to y).

Just for your information, there are connected spaces that are not path-connected. A famous example is "the topologist's sine curve" where X consists of all points on the graph $y = \sin \frac{1}{x}$, $x \neq 0$, pluss the point $(0, 0)$, and the metric is the one inherited from $\mathbb{R}^2$.

English-Norwegian dictionary for those who want to write in Norwegian

set: mengde
subset: delmengde
sequence: følge
subsequence: delfølge
series: rekke
converge uniformly: konvergere uniformt
metric space: metrisk rom
boundary point: randpunkt
closure: tillukning
precompact: prekompakt
connected: sammenhengende
disconnected: usammenhengende
path-connected: veisammenhengende

See http://folk.uio.no/klara/ordliste/index.html for more