MAT2410 - MANDATORY ASSIGNMENT #1, FALL 2010; FASIT

REMINDER: The assignment must be handed in before 14:30 on Thursday September 23, 2010, at the reception of the Department of Mathematics, in the 7th floor of Niels Henrik Abels hus, Blindern. Be careful to give reasons for your answers. To have a passing grade you must have correct answers to at least 50% of the questions and moreover have attempted to solve all of them.

Exercise 1.

a. Determine

$$\operatorname{Arg}(-6-6i), \quad \operatorname{Arg}(-\pi), \quad \operatorname{Arg}(3i), \quad \operatorname{Arg}(\sqrt{3}-i).$$

Answer. The principal arguments are $\theta_1 = -\frac{3\pi}{4}, \theta_2 = \pi, \theta_3 = \frac{\pi}{2}, \theta_4 = -\frac{\pi}{6}$.

b. Express the complex numbers

$$z_1 = -6-6i, \quad z_2 = -\pi, \quad z_3 = 3i, \quad z_4 = \sqrt{3} - i$$

in polar form, and compute

$$z_1 z_2 z_3 z_4.$$

Answer. Let r_j denote the modulus of z_j, i.e.,

$$r_1 = 6\sqrt{2}, \quad r_2 = \pi, \quad r_3 = 3, \quad r_4 = 2.$$

Then

$$z_j = r_j(\cos(\operatorname{Arg}(z_j) + 2\pi k) + i\sin(\operatorname{Arg}(z_j) + 2\pi k)), \qquad k = 0, \pm 1, \pm 2, \ldots,$$

for $j = 1, 2, 3, 4$.

Moreover,

$$z_1 z_2 z_3 z_4 = \rho(\cos(\theta + 2\pi k) + i\sin(\theta + 2\pi k)), \qquad k = 0, \pm 1, \pm 2, \ldots,$$

where

$$\rho = r_1 r_2 r_3 r_4 = 36\sqrt{2}\pi, \qquad \theta = \theta_1 + \theta_2 + \theta_3 + \theta_4 = \frac{7\pi}{12}.$$

c. Describe the set of points in the complex plane that satisfy

$$z\bar{z} = \frac{1}{2}(z + \bar{z})^2 + 1.$$

Date: October 5, 2010.

Answer. This set of points can be expressed as

$$|z|^2 = 2(\mathrm{Re}(z))^2 + 1.$$

With $z = x + iy$ this becomes $x^2 + y^2 = 2x^2 + 1$ or $y^2 = x^2 + 1$ or $y = \pm\sqrt{x^2+1}$.

d. Write e^{e^i} in standard form, i.e., as $a + bi$ for some real numbers a, b.

Answer. $e^{e^i} = e^{\cos(1)}\,[\cos(\sin(1)) + i\sin(\sin(1))]$.

Exercise 2.

a. Suppose z is a complex number different from 1 and $n \geq 1$ is an integer. Show that

$$1 + z + z^2 + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1}. \tag{1}$$

Answer. A computation shows that

$$(z-1)(1 + z + z^2 + \cdots + z^n) = z^{n+1} - 1,$$

which implies the result.

b. Utilize (1) and De Moivre's formula to show that

$$\sin(\theta) + \sin(2\theta) + \cdots + \sin(n\theta) = \frac{\sin(n\theta/2)\sin((n+1)\theta/2)}{\sin(\theta/2)}, \qquad \theta \in [0, 2\pi].$$

Answer. Take $z = e^{i\theta}$ with $\theta \neq 0$. Then the left-hand side of (1) becomes

$$\begin{aligned} 1 + z + z^2 + \cdots + z^n &= 1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} \\ &= (1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta)) \\ &\quad + i\,(\sin(\theta) + \sin(2\theta) + \cdots + \sin(n\theta)) \qquad \text{(De Moivre).} \end{aligned}$$

On the other hand, the right-hand side of (1) can be written

$$\begin{aligned} \frac{z^{n+1} - 1}{z - 1} &= \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1} \\ &= \frac{[\cos((n+1)\theta) - 1] + i\sin((n+1)\theta)}{[\cos(\theta) - 1] + i\sin(\theta)} \frac{[\cos(\theta) - 1] - i\sin(\theta)}{[\cos(\theta) - 1] - i\sin(\theta)} \\ &= \frac{[\cos((n+1)\theta) - 1] + i\sin((n+1)\theta)}{[\cos(\theta) - 1]^2 + \sin^2(\theta)} ([\cos(\theta) - 1] - i\sin(\theta)) \\ &= \frac{([\cos((n+1)\theta) - 1] + i\sin((n+1)\theta))([\cos(\theta) - 1] - i\sin(\theta))}{2 - 2\cos(\theta).} \end{aligned}$$

Observe that

$$\begin{aligned} &([\cos((n+1)\theta) - 1] + i\sin((n+1)\theta))([\cos(\theta) - 1] - i\sin(\theta)) \\ &= [\cos((n+1)\theta) - 1][\cos(\theta) - 1] + \sin((n+1)\theta)\sin(\theta) \\ &\qquad + i\,(-[\cos((n+1)\theta) - 1]\sin(\theta) + [\cos(\theta) - 1]\sin((n+1)\theta)) \\ &=: a + ib, \end{aligned}$$

Next,

$$\begin{aligned}
a &= \cos((n+1)\theta)\cos(\theta) - \cos((n+1)\theta) - \cos(\theta) + 1 + \sin((n+1)\theta)\sin(\theta)\\
&= \cos(n\theta) - \cos((n+1)\theta) - \cos(\theta) + 1\\
&\qquad \text{(we used } \cos(\theta_1-\theta_2) = \cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2))\\
&= 2\sin((n+1/2)\theta)\sin(\theta/2) + 1 - \cos(\theta)\\
&\qquad \text{(we used } \cos(\theta_1) - \cos(\theta_2) = -2\sin((\theta_1+\theta_2)/2)\sin((\theta_1-\theta_2)/2)
\end{aligned}$$

and, since also $1 - \cos(\theta) = \cos(0) - \cos(\theta) = 2\sin^2(\theta/2)$,

$$\frac{a}{2-2\cos(\theta)} = \frac{2\sin((n+1/2)\theta)\sin(\theta/2) + 2\sin^2(\theta/2)}{4\sin^2(\theta/2)} = \frac{\sin((n+1/2)\theta)}{2\sin(\theta/2)} + \frac{1}{2}.$$

Similarly,

$$\begin{aligned}
b &= -\cos((n+1)\theta)\sin(\theta) + \sin(\theta) + \cos(\theta)\sin((n+1)\theta) - \sin((n+1)\theta)\\
&= \sin(n\theta) + \sin(\theta) - \sin((n+1)\theta)\\
&\qquad \text{(we used } \sin(\theta_1-\theta_2) = \sin(\theta_1)\cos(\theta_2) - \sin(\theta_2)\cos(\theta_1))\\
&= -2\sin(\theta/2)\cos((2n+1)\theta/2) + 2\sin(\theta/2)\cos(\theta/2)\\
&\qquad \text{(we used } \sin(\theta_1) - \sin(\theta_2) = 2\sin((\theta_1-\theta_2)/2)\cos((\theta_1+\theta_2)/2) \text{ and } \sin(\theta) = 2\sin(\theta/2)\cos(\theta/2)\\
&= 2\sin(\theta/2)\Big[\cos(\theta/2) - \cos((2n+1)\theta/2)\Big]\\
&= 2\sin(\theta/2)\Big[-2\sin((\theta/2 - (2n+1)\theta/2)/2)\sin((\theta/2 + (2n+1)\theta/2)/2)\Big]\\
&\qquad \text{(we used } \cos(\theta_1) - \cos(\theta_2) = -2\sin((\theta_1+\theta_2)/2)\sin((\theta_1-\theta_2)/2)\\
&= 2\sin(\theta/2)\Big[-2\sin(-n\theta/2)\sin((n+1)\theta/2)\Big]\\
&= 4\sin(\theta/2)\sin(n\theta/2)\sin((n+1)\theta/2),
\end{aligned}$$

and hence

$$\frac{b}{2-2\cos(\theta)} = \frac{b}{4\sin^2(\theta/2)} = \frac{\sin(n\theta/2)\sin((n+1)\theta/2)}{\sin(\theta/2)}.$$

Summarizing, we have proved

$$\frac{z^{n+1}-1}{z-1} = \left(\frac{\sin((n+1/2)\theta)}{2\sin(\theta/2)} + \frac{1}{2}\right) + i\left(\frac{\sin(n\theta/2)\sin((n+1)\theta/2)}{\sin(\theta/2)}\right),$$

which, from our previous calculation, equals

$$\begin{aligned}
&(1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta))\\
&\quad + i\,(\sin(\theta) + \sin(2\theta) + \cdots + \sin(n\theta))\,.
\end{aligned}$$

Thus, the result follows from comparing the imaginary parts of the two expressions.

c. Similarly, show that

$$1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta) = \frac{1}{2} + \frac{\sin((n+1/2)\theta)}{2\sin(\theta/2)}.$$

Answer. Use the previous result, this time comparing real parts.

d. Compute all the values of

$$(1-i)^{5/2}.$$

Plot these values in the complex plane.

Answer. First compute

$$(1-i)^{1/2} = (\sqrt{2})^{1/2} e^{i(-\pi/4+2\pi k)/2}, \qquad k = 0, 1,$$

and then

$$\left((1-i)^{1/2}\right)^5 = 2^{5/4} e^{5i(-\pi/4+2\pi k)/2} = 2^{5/4} e^{i(-5\pi/8+5\pi k)}, \qquad k = 0, 1.$$

Hence

$$(1-i)^{5/2} = \begin{cases} 2^{5/4} e^{-\frac{5\pi}{8} i}, \\ 2^{5/4} e^{\frac{3\pi}{8} i}. \end{cases}$$

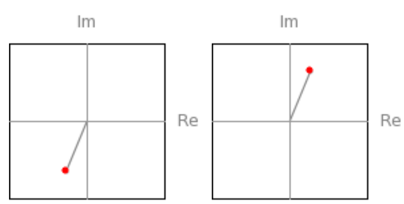

FIGURE 1. Left: $2^{5/4} e^{-\frac{5\pi}{8} i}$. Right: $2^{5/4} e^{\frac{3\pi}{8} i}$.

Exercise 3.

a. Consider the Riemann sphere

$$\Sigma = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\},$$

and the subset

$$S = \{(x_1, x_2, x_3) \in \Sigma : x_3 = 1/2\}.$$

Describe the set in the complex plane that under stereographic projection maps to the set S on the Riemann sphere.

Answer. Let $z = x + iy$ be a point in the complex plane. We have (cf. the book)

$$x = \frac{x_1}{1 - x_3}, \qquad y = \frac{x_2}{1 - x_3},$$

which yields

$$x^2 + y^2 = 4(x_1^2 + x_2^2) = 4(1 - x_3^2) = 4(1 - 1/4) = 3;$$

in other words, a circle in the complex plane with radius $\sqrt{3}$.

b. Let z be a complex number with $|z| < 1$. For $n = 1, 2, \ldots$, set

$$S_n = 1 + z + z^2 + \cdots + z^n.$$

Show that

$$\lim_{n\to\infty} S_n = \frac{1}{1 - z}.$$

Answer. Use (1) and the fact that $|z|^{n+1} \to 0$ as $n \to \infty$.