MAT2410 - MANDATORY ASSIGNMENT #2, FALL 2010; FASIT

REMINDER: The assignment must be handed in before 14:30 on Thursday October 28 at the Department of Mathematics, in the 7th floor of Niels Henrik Abels hus, Blindern. Be careful to give reasons for your answers. To have a passing grade you must have correct answers to at least 50% of the questions.

Exercise 1.

a. Consider the function

$$f(z) = (x^3 + 3xy^2 - 3x) + i\,(y^3 + 3x^2y - 3y).$$

Determine the points $z = x + iy$ at which $f(z)$ is differentiable.

Answer. Set $u(x, y) = x^3 + 3xy^2 - 3x$ and $v(x, y) = y^3 + 3x^2y - 3y$. We compute

$$\frac{\partial}{\partial x}u = 3x^2 + 3y^2 - 3 = \frac{\partial}{\partial y}v,$$

but

$$\frac{\partial}{\partial y}u = 6xy = \frac{\partial}{\partial x}v,$$

and thus $\frac{\partial}{\partial y}u = -\frac{\partial}{\partial x}v$ if and only if $x = 0$ or $y = 0$, i.e., by the Cauchy-Riemann equations $f(z) = u(x, y) + i\,v(x, y)$ is differentiable at the coordinate axes.

b. Let $f(z) = u(x, y) + i\,v(x, y)$ be a function that is analytic in a domain $D \subset \mathbb{C}$, and suppose the real part of f, $\mathrm{Re}(f)$, is constant in D. Show that then f must be constant in D.

Answer. The function $f(z) = u(x, y) + i\,v(x, y)$ is differentiable at a point z implies that the Cauchy-Riemann equations hold at that point:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

By assumption, $\frac{\partial}{\partial x}u = 0$ and $\frac{\partial}{\partial y}u = 0$; thus $\frac{\partial}{\partial x}v = 0$ (and $\frac{\partial}{\partial y}v = 0$). Consequently,

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = 0 \quad \forall z \in D,$$

and by Theorem 6 (p. 76) this imples $f =$ constant in D.

Date: November 4, 2010.

c. Let $f(z)$ be a function such that both $f(z)$ and $\overline{f(z)}$ are analytic in a domain $D \subset \mathbb{C}$. Show that then f must be constant in D.

Answer. $\mathrm{Re}(f) = \frac{f+\overline{f}}{2}$ is (real-valued) analytic if and only if f and $\overline{f}$ are analytic. If $f = u + i\,v$, then $\overline{f} = u - i\,v$. These functions are differentiable at a point z if the Cauchy-Riemann equations hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (\text{for } f);$$

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \quad (\text{for } \overline{f}).$$

Adding these equations yields

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0,$$

i.e., the real part of f is constant. The result follows form b).

d. Determine the poles of the function

$$\frac{z^4 + 2z^3}{z^5 + (2+2i)z^4 + 4iz^3}.$$

Answer. First, write the polynomial in the denominator $z^5 + (2+2i)z^4 + 4iz^3$ in factored form. Real roots $z = -2$ and $z = 0$ (multiplicity 3). Complex root $z = -2i$. Factored form is $z^3(z+2)(z+2i)$. Next, write $z^4 + 2z^3 = z^3(z+2)$. Hence

$$\frac{z^4 + 2z^3}{z^5 + (2+2i)z^4 + 4iz^3} = \frac{z^3(z+2)}{z^3(z+2)(z+2i)} = \frac{1}{z+2i}.$$

i.e, the (single) pole is $z = -2i$.

Exercise 2.

a. Consider the function $f(z) = \cos^2(z) + \sin^2(z)$. Prove that this function is entire and $f'(z) = 0$ for all z. Use this to conclude that $\sin^2(z) + \cos^2(z) = 1$ for each z.

Answer. The functions $\sin(z)$ and $\cos(z)$ are entire, and the chain rule implies that $\cos^2(z)$ and $\sin^2(z)$ are entire functions. Moreover, $f'(z) = -2\cos(z)\sin(z) + 2\sin(z)\cos(z) = 0$. Hence, $f(z) = C$ for some constant C. We determine C by $f(0) = \cos^2(0) + \sin^2(0) = 1 + 0 = 1$, so $C = 1$.

b. Determine the domain D on which $f(z) = \mathrm{Log}(4+i-z)$ is analytic. Draw a picture of D in the complex plane. Compute $f'(z)$. What is the domain of analyticity of $\mathcal{L}_{\frac{\pi}{2}}(z)$, where $\mathcal{L}_{\frac{\pi}{2}}$ refers to the branch of $\log z$ based on the cut along the positive imaginary axis (draw a picture of the domain). Compute $\frac{d}{dz}\mathcal{L}_0(z)$.

Answer. Domain of analyticity of $\mathrm{Log}(4 + i - z)$ is

$$\mathbb{C} \setminus \{z = x + iy : x \geq 4, y = 1\},$$

and $\frac{d}{dz}\mathrm{Log}(4 + i - z) = \frac{-1}{4+i-z}$.

Domain of analyticity of $\mathcal{L}_{\frac{\pi}{2}}(z)$ is

$$\mathbb{C} \setminus \{z = x + iy : x = 0, y \geq 0\},$$

and $\frac{d}{dz}\mathcal{L}_0(z) = \frac{1}{z}$.

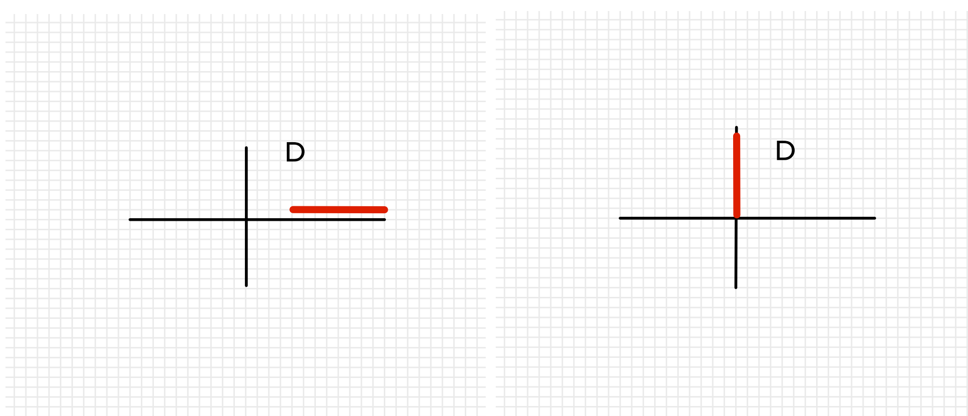

FIGURE 1. Left. Domain of analyticity of $\mathrm{Log}(4 + i - z)$. Right. Domain of analyticity of $\mathcal{L}_{\frac{\pi}{2}}(z)$.

c. Consider the set (annulus) $D = \{z \in \mathbb{C} : 2 < |z - 1| < 4\}$. Prove that there does not exist a function $f(z)$ that is analytic on D such that $f'(z) = \frac{1}{z}$ for all $z \in D$. [Hint: Assume that such a function exists and then show that this leads to a contradiction.]

Answer. If such a function exists, set

$$h(z) = f(z) - \mathrm{Log}(z).$$

Then $h'(z) = 0$ for all $z \in D \backslash [-3, -1]$. Consequently, h is constant in $D \backslash [-3, -1]$ and therefore $f(z) = \mathrm{Log}(z) + C$ for some constant C, in $D \setminus [-3, -1]$. This contradicts the assumption that f analytic in D.

d. Consider the function $\sec(z) = \frac{1}{\cos(z)}$. Derive the expression

$$\sec^{-1}(z) = -i \log \left(\frac{1}{z} + \left(\frac{1}{z^2} - 1 \right)^{\frac{1}{2}} \right)$$

for the inverse of $\sec(z)$. Compute the principal value $\mathrm{Sec}^{-1}(-1)$.

Answer. Set $w = \sec^{-1}(z)$. Then $z = \sec(w) = \frac{2}{e^{iw} + e^{-iw}}$ or

$$ze^{iw} + ze^{-iw} = 2 \quad \text{or} \quad ze^{i2w} + z - 2e^{iw} = 0.$$

Setting $W = e^{iw}$, we obtain the quadratic equation

$$zW^2 - 2W + z = 0 \Longrightarrow W = \frac{1}{z} + \left(\frac{1}{z^2} - 1 \right)^{\frac{1}{2}}.$$

In other words,

$$e^{iw} = \frac{1}{z} + \left(\frac{1}{z^2} - 1 \right)^{\frac{1}{2}}$$

and so

$$iw = \log \left(\frac{1}{z} + \left(\frac{1}{z^2} - 1 \right)^{\frac{1}{2}} \right),$$

which, after multiplying both sides by i, yields

$$\sec^{-1}(z) = -i \log \left(\frac{1}{z} + \left(\frac{1}{z^2} - 1 \right)^{\frac{1}{2}} \right).$$

Next, we compute the principal value $\mathrm{Sec}^{-1}(-1)$:

$$\mathrm{Sec}^{-1}(-1) = -i \, \mathrm{Log}(-1) = -i \left(\mathrm{Log}|-1| + i\mathrm{Arg}(-1) \right) = \mathrm{Arg}(-1) = \pi.$$

Exercise 3.

a. Determine an admissible parametrization $z(t)$, $t \in [0,3]$, of the contour Γ shown in Figure 2, which has initial point 0 and terminal point 1.

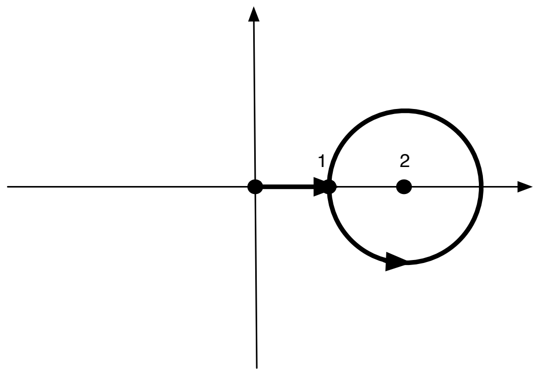

FIGURE 2. Contour Γ for Exercise 3.

Answer. A parameterization of Γ reads

$$z(t) = \begin{cases} t, & t \in [0,1], \\ 2 + e^{i\pi t}, & t \in [1,3]. \end{cases}$$

b. Compute the length of the contour shown in Figure 2, using the parameterization from a) and formula (1) on page 159 in the book.

Answer. The length $\ell(\Gamma) = \int_0^1 |z'(t)|\, dt + \int_1^3 |z'(t)|\, dt$ is $1 + 2\pi$, since

$$\int_0^1 \left| \frac{d}{dt} t \right| \, dt = 1$$

and

$$\int_1^3 \left| \frac{d}{dt} \left(2 + e^{i\pi t}\right) \right| \, dt = 2\pi.$$

c. Compute the integral $\int_\Gamma \frac{1}{z+1}\, dz$, where Γ is the contour shown in Figure 2.

Answer. The contour Γ consists of two parts (γ_1, γ_2), where $z_1(t) = t$, $t \in [0,1]$, and $z_2(t) = 2 + e^{i\pi t}$, $t \in [1,3]$. Hence

$$\int_\Gamma \frac{1}{z+1}\,dz = \int_{\gamma_1} \frac{1}{z+1}\,dz + \int_{\gamma_2} \frac{1}{z+1}\,dz.$$

First, $\frac{1}{z+1}$ has an antiderivative, namely the principal value of $\log(z+1)$:

$$\text{Log}(z+1) = \text{Log}|z+1| + i\,\text{Arg}(z+1).$$

Since γ_2 is a closed contour, it thus follows that (consult Corollary 2, page 175 in the book)

$$\int_{\gamma_2} \frac{1}{z+1}\,dz = 0.$$

Finally,

$$\begin{aligned}\int_{\gamma_1} \frac{1}{z+1}\,dz = \Big[\text{Log}(z+1)\Big]_0^1 &= \text{Log}(2) - \text{Log}(1) \\ &= (\text{Log}|2| + i\,\text{Arg}(2)) - (\text{Log}|1| + i\,\text{Arg}(1)) \\ &= (0.301 + i\,0) - (0 + i\,0) = 0.301.\end{aligned}$$

Therefore,

$$\int_\Gamma \frac{1}{z+1}\,dz = 0.301.$$