MAT2410: Mandatory assignment #2, autumn 2013

To be handed in by October 24., 14:30

Exercise 1.

Let $g : \mathbb{C} \to \mathbb{C}$ be defined by

$$g(z) = \begin{cases} \frac{z}{e^z - 1} & z \neq 0, \\ 1 & z = 0. \end{cases}$$

(**a**) Is g continuous in the disc $|z| < 2\pi$?
(**b**) Show that $\cot(z) = i + g(2iz)/z$, and hence that $g(z) = (z/2i)\cot(z/2i) - z/2$, and also that $z \mapsto g(z) + z/2$ is even.
(**c**) Find $g'(0)$ and show that g is analytic for $|z| < 2\pi$. **Hint**: Use L'Hopital's rule.
(**d**) We write the Taylor series for g around zero as

$$g(z) = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

The numbers B_n are called the *Bernoulli numbers*. We have that $B_0 = 1$ and $B_1 = -1/2$, why? In this notation

$$g(z) + \frac{z}{2} = 1 + \sum_{n=2}^{\infty} \frac{B_n}{n!} z^n.$$

Why does this imply that $B_{2n+1} = 0$ for all $n \geq 1$?
(**e**) Derive the relation

$$\binom{n}{0} B_0 + \binom{n}{1} B_1 + \binom{n}{2} B_2 + \cdots + \binom{n}{n-2} B_{n-2} + \binom{n}{n-1} B_{n-1} = 0,$$

where $\binom{n}{j} = n!/((n-j)!j!)$ are the binomial numbers.
Hint: use that $1 = ((e^z - 1)/z)g(z)$ and develop the Taylor series for the product.
(**f**) Find B_2 and B_4? Are the Bernoulli numbers bounded?

Exercise 2.

(**a**) Find all complex numbers w such that $w^4 = -1$.
(**b**) Find a branch of $(1 - z^4)^{1/4}$ which is analytic outside the unit circle centered at 0. **Hint**: Let w be as in **a**, write $1 - z^4 = (wz)^4 - w^4$.

Exercise 3.

Let γ be the circle $|z| = 2$, traversed once in the counterclockwise direction. Determine the integral

$$\int_\gamma \frac{\cos(z)}{1 - z^3} \, dz.$$