MAT2410: Mandatory assignment #1, autumn 2016

To be handed in by September 29, 14:30

Exercise 1.

(**a**) Find all solutions of

$$z^6 = \frac{1+i}{\sqrt{3}+i}.$$

Possible solution: We have that $(1+i)/(\sqrt{3}+i) = (1/\sqrt{2})e^{i\pi/12}$. Hence the roots are

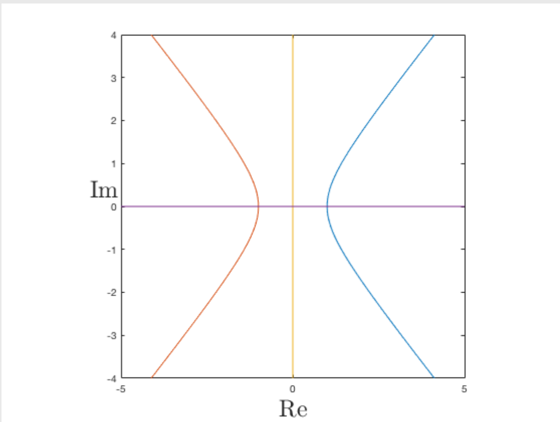
$$z = \frac{1}{\sqrt[12]{2}} e^{i\frac{\pi}{12}} e^{ik\frac{\pi}{3}}, \quad k = 0, 1, \ldots, 5.$$

(**b**) Find the two values of $(7+24i)^{1/2}$.

Possible solution: By trial & error we observe that $7+24i = (4+3i)^2$, thus the two roots are $\pm(4+3i)$.

(**c**) Sketch the set $\{z \,|\, z^2 + \bar{z}^2 = 2\}$.

Possible solution: Let $z = x+iy$, we have that $z^2 + \bar{z}^2 = 2\,\mathrm{Re}(z^2)$, thus we must have $x^2 - y^2 = 1$, or $y = \pm\sqrt{x^2-1}$, which are hyperbola.



(**d**) Solve the equation

$$z^2 - (1+i)z + \frac{1}{4}i = 0.$$

Possible solution:

$$\begin{aligned} z &= \frac{1}{2}(1+i+\sqrt{i}), \qquad (\,\sqrt{i} = \pm e^{i\pi/4} = \pm(1+i)/\sqrt{2}\,) \\ &= \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{2}}\right)(1+i). \end{aligned}$$

(**e**) Let $f(z) = \alpha z + \beta$, where α and β are complex numbers. Prove that if $|\alpha| = 1$, then $|z_1 - z_2| = |f(z_1) - f(z_2)|$.

Possible solution:

$$f(z_1) - f(z_2) = \alpha(z_1 - z_2),$$

hence $|f(z_1) - f(z_2)| = |\alpha|\,|z_1 - z_2| = |z_1 - z_2|$ if $|\alpha| = 1$.

Exercise 2.

Recall that a set $U \subseteq \mathbb{C}$ is called *convex* if the straight line between any two points z_1 and z_2 in U also lies in U, i.e., we have for $t \in [0,1]$,

$$tz_1 + (1-t)z_2 \in U.$$

Show that

(**a**) If U and V are convex, then $U \cap V$ is convex.

Possible solution: Choose z_1 and z_2 in $U \cap V$, and write $\gamma(t) = tz_1 + (1-t)z_2$, we have that since U is convex, $\gamma(t) \in U$ and since V is convex, $\gamma(t) \in V$. Hence, $\gamma(t) \in U \cap V$.

(**b**) If U is convex, then also $U \cup \partial U$ is convex. (You can use that $z \in U \cup \partial U$ iff there is a sequence $\{w_k\} \subset U$ such that $w_k \to z$.)

Possible solution: We find sequences $\{\omega_{k,1}\}$ and $\{\omega_{k,2}\}$ such that $\omega_{k,j} \in U$ for $j = 1,2$ and $k \geq 1$ and $\omega_{k,j} \to z_j$. Write $\gamma(\alpha,\beta,t) = t\alpha + (1-t)\beta$ for $\alpha, \beta \in \mathbb{C}$. We have that γ is continuous in all variables. Fix t, and consider the sequence $\{\gamma_k\} = \{\gamma(\omega_{k,1}, \omega_{k,2}, t)\}$. This sequence is a Cauchy sequence since

$$\begin{aligned} |\gamma_k - \gamma_l| &= |\gamma(\omega_{k,1}, \omega_{k,2}, t) - \gamma(\omega_{l,1}, \omega_{l,2}, t)| \\ &= |t(\omega_{k,1} - \omega_{l,1}) + (1-t)(\omega_{k,2} - \omega_{l,2})| \\ &\leq t\,|\omega_{k,1} - \omega_{l,1}| + (1-t)\,|\omega_{k,2} - \omega_{l,2}| \\ &\leq \max\{|\omega_{k,1} - \omega_{l,1}|\,, |\omega_{k,2} - \omega_{l,2}|\}\,. \end{aligned}$$

Hence γ_k converges to a point in $U \cup \partial U$.

Exercise 3.

Let the (Euclidean) inner product between two complex numbers z and w be defined as

$$\langle z, w\rangle = \mathrm{Re}(z\bar{w}).$$

(**a**) Show that $|z| = \langle z, z\rangle^{1/2}$, the Cauchy-Schwartz inequality $|\langle z, w\rangle| \leq |z|\,|w|$ and the triangle inequality.

Possible solution: The norm follows by expanding $\mathrm{Re}(z\bar{w})$. Since we have $|\mathrm{Re}(z)| = |\mathrm{Re}(\bar{z})| \leq |z|$, we get $|\langle z, w\rangle| \leq |z|\,|w|$. For the triangle inequality, we have

$$|z+w|^2 = |z|^2 + 2\langle z, w\rangle + |w|^2 \leq |z|^2 + |w|^2\,.$$

(**b**) Show that $\langle z, w\rangle = 0$ iff z/w is pure imaginary (has zero real part).

Possible solution: We have that $\bar{w} = |w|/w$, and thus

$$\langle z, w\rangle = \mathrm{Re}(z\bar{w}) = \mathrm{Re}(|w|\; z/w) = |w|\,\mathrm{Re}(z/w).$$

Exercise 4.

Find the limits

(**a**)

$$\lim_{n\to\infty} \left(\frac{i}{2}\right)^n,$$

Possible solution: 0

(**b**)

$$\lim_{z\to -i} \frac{z^2+1}{z+i},$$

Possible solution: $-2i$

(**c**)

$$\lim_{n\to\infty} \left(1+\frac{i}{n}\right)^{n\pi}.$$

Possible solution: $e^{i\pi} = -1$

Exercise 5.

Let $z = x + iy$ and $f = f(z) = u(x, y) + iv(x, y)$. We define the "partial derivatives"

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y},$$
$$\frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}).$$

(**a**) Find $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$, for the following functions: i) $f(z) = z^2$, ii) $f(z) = e^z$, iii) $f(z) = |z|^2$, iv) $f(z) = \mathrm{Im}(z)$.

Possible solution:
i) $f_z = 2z$, $f_{\bar{z}} = 0$,
ii) $f_z = e^z$, $f_{\bar{z}} = 0$,
iii) $f(z) = z\bar{z}$, so that $f_z = \bar{z}$, $f_{\bar{z}} = z$.
iv) $f(z) = (z - \bar{z})/2i$, so that $f_z = 1/2i$, $f_{\bar{z}} = -1/(2i)$.

(**b**) Show that inverse formulas

$$\frac{\partial u}{\partial x} = \frac{1}{2}(\frac{\partial f}{\partial x} + \overline{\frac{\partial f}{\partial x}}), \quad \frac{\partial u}{\partial y} = \frac{1}{2}(\frac{\partial f}{\partial y} + \overline{\frac{\partial f}{\partial y}}),$$
$$\frac{\partial v}{\partial x} = \frac{1}{2i}(\frac{\partial f}{\partial x} - \overline{\frac{\partial f}{\partial x}}), \quad \frac{\partial v}{\partial y} = \frac{1}{2i}(\frac{\partial f}{\partial y} - \overline{\frac{\partial f}{\partial y}}),$$
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}, \quad \frac{\partial f}{\partial y} = i(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}).$$

Possible solution: This follows by inverting the formulas above.

(**c**) Let $D(f) = \frac{\partial f}{\partial z}$ or $D(f) = \frac{\partial f}{\partial \bar{z}}$. Show that $D(\alpha f + \beta g) = \alpha D(f) + \beta D(g)$ for any complex constants α and β, and that the Leibnitz and quotient rules hold:

$$D(fg) = D(f)g + fD(g), \quad D(f/g) = (D(f)g - fD(g))/g^2.$$

Possible solution: D is a linear combination of operators for which these rules hold.

We can now view a complex function f as a function of z and $\bar{z}$ instead of (x, y).

(**d**) Show that the Cauchy-Riemann equations hold iff $\frac{\partial f}{\partial \bar{z}} = 0$, and that in this case $f'(z) = \frac{\partial f}{\partial z}$.

Possible solution:

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{i}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}) \\ &= \frac{i}{2}\left((\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) + i(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})\right),\end{aligned}$$

so CR hold iff $f_{\bar{z}} = 0$. We also have that if f is analytic,

$$\begin{aligned}f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + \overline{\frac{\partial f}{\partial x}} + \frac{\partial f}{\partial x} - \overline{\frac{\partial f}{\partial x}}\right) \\ &= \frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial z},\end{aligned}$$

since $f_{\bar{z}} = 0$.

(**e**) Use this to determine if the following functions are analytic: i) $\mathrm{Re}(z)$, ii) $(x^2 - y^2) + 2xyi$, iii) e^{iy}, iv) $z(z + \bar{z}^2)$.

Possible solution:

i) $\mathrm{Re}(z) = (z + \bar{z})/2$, not analytic.
ii) $f(z) = z^2$, analytic.
iii) $f(z, \bar{z}) = e^{(z-\bar{z})/2}$, not analytic.
iv) $\frac{\partial f}{\partial \bar{z}} = 2z\bar{z}$, not differentiable for $\bar{z} \neq 0$.

Exercise 6.

Let $f(z)$ be defined as

$$f(z) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} + i\frac{x^3+y^3}{x^2+y^2} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

(**a**) Show that f is continuous at $z = 0$. Hint: You can use that $|x^3 \pm y^3| \leq 2(|x| + |y|)(x^2 + y^2)$.

Possible solution:

$$\begin{aligned}\lim_{z\to 0}|f(z)| &\le \lim_{(x,y)\to(0,0)} \left|\frac{x^3-y^3}{x^2+y^2}\right| + \left|\frac{x^3+y^3}{x^2+y^2}\right| \\ &\le 4 \lim_{(x,y)\to(0,0)} |x|+|y| = 0 = f(0).\end{aligned}$$

(**b**) Is f analytic?

Possible solution:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{x(x^3+3xy^2+2y^3)}{(x^2+y^2)^2}, \\ \frac{\partial u}{\partial y} &= -\frac{y(2x^3+3x^2y+y^3)}{(x^2+y^2)^2}, \\ \frac{\partial v}{\partial x} &= \frac{x(x^3+3xy^2-2y^3)}{(x^2+y^2)^2}, \\ \frac{\partial v}{\partial y} &= \frac{y(-2x^3+3x^2y+y^3)}{(x^2+y^2)^2}\end{aligned}$$

We have that

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 4\frac{xy^3}{(x^2+y^2)^2} \neq 0, \quad \text{for } x \neq 0,\ y \neq 0.$$

For $(x, y) = (0, 0)$, $\lim(u_x - v_y)$ can be any number in $[-1, 1]$. Hence f is not analytic.