MAT2410: Mandatory assignment #1, autumn 2016

To be handed in by September 29, 14:30

Exercise 1.

(**a**) Find all solutions of

$$z^6 = \frac{1+i}{\sqrt{3}+i}.$$

(**b**) Find the two values of $(7+24i)^{1/2}$.

(**c**) Sketch the set $\{z \mid z^2 + \bar{z}^2 = 2\}$.

(**d**) Solve the equation

$$z^2 - (1+i)z + \frac{1}{4}i = 0.$$

(**e**) Let $f(z) = \alpha z + \beta$, where α and β are complex numbers. Prove that if $|\alpha| = 1$, then $|z_1 - z_2| = |f(z_1) - f(z_2)|$.

Exercise 2.

Recall that a set $U \subseteq \mathbb{C}$ is called *convex* if the straight line between any two points z_1 and z_2 in U also lies in U, i.e., we have for $t \in [0,1]$,

$$tz_1 + (1-t)z_2 \in U.$$

Show that

(**a**) If U and V are convex, then $U \cap V$ is convex.

(**b**) If U is convex, then also $U \cup \partial U$ is convex. (You can use that $z \in U \cup \partial U$ iff there is a sequence $\{w_k\} \subset U$ such that $w_k \to z$.)

Exercise 3.

Let the (Euclidean) inner product between two complex numbers z and w be defined as

$$\langle z, w \rangle = \mathrm{Re}(z\bar{w}).$$

(**a**) Show that $|z| = \langle z, z \rangle^{1/2}$, the Cauchy-Schwartz inequality $|\langle z, w \rangle| \leq |z|\,|w|$ and the triangle inequality.

(**b**) Show that $\langle z, w \rangle = 0$ iff z/w is pure imaginary (has zero real part).

Exercise 4.

Find the limits

(**a**)

$$\lim_{n\to\infty}\left(\frac{i}{2}\right)^n,$$

(**b**)

$$\lim_{z\to -i}\frac{z^2+1}{z+i},$$

(**c**)

$$\lim_{n\to\infty}\left(1+\frac{i}{n}\right)^{n\pi}.$$

Exercise 5.

Let $z = x + iy$ and $f = f(z) = u(x, y) + iv(x, y)$. We define the "partial derivatives"

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y},$$

$$\frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}).$$

(**a**) Find $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$, for the following functions: i) $f(z) = z^2$, ii) $f(z) = e^z$, iii) $f(z) = |z|^2$, iv) $f(z) = \mathrm{Im}(z)$.

(**b**) Show that inverse formulas

$$\frac{\partial u}{\partial x} = \frac{1}{2}(\frac{\partial f}{\partial x} + \overline{\frac{\partial f}{\partial x}}), \quad \frac{\partial u}{\partial y} = \frac{1}{2}(\frac{\partial f}{\partial y} + \overline{\frac{\partial f}{\partial y}}),$$

$$\frac{\partial v}{\partial x} = \frac{1}{2i}(\frac{\partial f}{\partial x} - \overline{\frac{\partial f}{\partial x}}), \quad \frac{\partial v}{\partial y} = \frac{1}{2i}(\frac{\partial f}{\partial y} - \overline{\frac{\partial f}{\partial y}}),$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}, \quad \frac{\partial f}{\partial y} = i(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}).$$

(**c**) Let $D(f) = \frac{\partial f}{\partial z}$ or $D(f) = \frac{\partial f}{\partial \bar{z}}$. Show that $D(\alpha f + \beta g) = \alpha D(f) + \beta D(g)$ for any complex constants α and β, and that the Leibnitz and quotient rules hold:

$$D(fg) = D(f)g + fD(g), \quad D(f/g) = (D(f)g - fD(g))/g^2.$$

We can now view a complex function f as a function of z and $\bar{z}$ instead of (x, y).

(**d**) Show that the Cauchy-Riemann equations hold iff $\frac{\partial f}{\partial \bar{z}} = 0$, and that in this case $f'(z) = \frac{\partial f}{\partial z}$.

(**e**) Use this to determine if the following functions are analytic: i) $\mathrm{Re}(z)$, ii) $(x^2 - y^2) + 2xyi$, iii) e^{iy}, iv) $z(z + \bar{z}^2)$.

Exercise 6.

Let $f(z)$ be defined as

$$f(z) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} + i\frac{x^3+y^3}{x^2+y^2} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

(**a**) Show that f is continuous at $z = 0$. Hint: You can use that $|x^3 \pm y^3| \leq 2(|x| + |y|)(x^2 + y^2)$.

(**b**) Is f analytic?