Exercises
(Translation from Sydsæter, Seierstad and Strøm's book)

Ex. 1 p. 341

Consider the problem of minimizing $\int_0^1 (t\dot{x} + \dot{x}^2)\,dt$, $x(0) = 1$, $x(1) = 0$.

a) Set up the Euler-Lagrange equation and solve it.

b) Find the only solution of the Euler-Lagrange equation which satisfies the boundary conditions.

Ex. 3 p. 341

Find the Euler-Lagrange equations for $J(x) = \int_{t_0}^{t_1} F(t, x, \dot{x})\,dt$ when

a) $F(t, x, \dot{x}) = x^2 + \dot{x}^2 + 2xe^t$

b) $F(t, x, \dot{x}) = -e^{\dot{x} - ax}$

c) $F(t, x, \dot{x}) = \left((x - \dot{x})^2 + x^2\right) e^{-at}$

d) $F(t, x, \dot{x}) = 2tx + 3x\dot{x} + t\dot{x}^2$

Ex. 6 p. 341

Consider the variational problem

$$\max \int_0^T U(\bar{c} - \dot{x}e^{rt}), \quad x(0) = x_0, \quad x(T) = 0$$

where $x = x(t)$ is the unknown function, T, $\bar{c}$, r and x_0 are positive constants and U is a given differentiable function.

a) Set up the Euler-Lagrange equation for this problem.

b) Solve the Euler-Lagrange equation when $U(c) = c^{1-v}/(1 - v)$ where v is a constant in $[0, 1]$.

Ex. 8 p. 341

Solve the variational problem

$$\min \int_0^1 (x^2 + tx + tx\dot{x} + \dot{x}^2)\,dt, \quad x(0) = 0, \quad x(1) = 1.$$

Show that the integrand is convex, that is, show that the function $F : \mathbb{R}^2 \to \mathbb{R}$ given by

$$F(x_1, x_2) = x_1^2 + tx_1 + x_1x_2 + x_2^2$$

is convex for all $t \in [0, 1]$.