SSS:

11.2 (1) Consider the problem:

$$\min_x \int_0^1 (t\dot{x} + \dot{x}^2)\,dt, \quad x(0)=1, \; x(1)=0$$

(a) Let $F(t, x, \dot{x}) = t\dot{x} + \dot{x}^2$

The Euler equation of the problem is

$$0 = \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 - \frac{d}{dt}\left(\underbrace{t + 2\dot{x}}\right)$$

$$\left[= -(1+2\ddot{x})\right]$$

or

$$\frac{d}{dt}(t + 2\dot{x}) = 0$$

$$t + 2\dot{x} = C_1'$$

$$\dot{x}(t) = C_1 - \frac{1}{2}t$$

$$\underbrace{x(t) = C_1 t - \frac{1}{4}t^2 + C_2},$$

(b) By the endpoint conditions

$$x(0) = C_2 = 1, \quad \underline{C_2 = 1}$$

$$x(1) = C_1 - \frac{1}{4} + 1 = 0, \quad \underline{C_1 = -\frac{3}{4}}$$

$$\underline{x(t) = -\frac{1}{4}t^2 - \frac{3}{4}t + 1}$$

(c) Here

$$\frac{\partial^2 F}{\partial x^2} = \underline{0} \geq 0 \quad \frac{\partial^2 F}{\partial \dot{x} \partial x} = \underline{0}$$

$$\frac{\partial^2 F}{\partial \dot{x}^2} = 2 > 0$$

$$\Delta_F = 0 \cdot 2 - 0^2 = 0 \geq \underline{0}.$$

Hence F is convex in $(x, \dot{x})$ for each fixed t in $[0,1]$.

Hence the solution x in (b) gives the minimum.

11.2 (7) Solve:

$$\min_x \int_1^2 (2tx + \underbrace{3x\dot{x}} + t\dot{x}^2)\,dt, \quad x(1)=0, \; x(2)=1.$$

Let $F(t, x, \dot{x}) = 2tx + 3x\dot{x} + t^2\dot{x}^2$

Here the 2nd-derivative test fails since

$$\Delta_F = \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial \dot{x} \partial y}\right)^2 = -9 < 0.$$

Here $\quad 3x\dot{x} = \frac{d}{dt}\left(\frac{3}{2}x^2\right)$

hence

$$\int_1^2 (2tx + 3x\dot{x} + t\dot{x}^2)\,dt = \int_1^2 (2tx + t\dot{x}^2)\,dt + \int_1^2 \frac{d}{dt}\left(\frac{3}{2}x^2\right)dt$$

$$= \int_1^2 (2tx + t\dot{x}^2)\,dt - \frac{3}{2}\cdot 0 + \frac{3}{2}\cdot 1^2$$

Hence it suffices to minimize

$$\int_1^2 (2tx + t\dot{x}^2)\,dt$$

Let $\quad G(t,x,\dot{x}) = 2tx + t\dot{x}^2$

(which is a sum of two convex functions, hence is convex).

$$\frac{\partial G}{\partial x} = 2t \qquad \frac{\partial^2 G}{\partial x^2} = 0 \geq 0, \quad \frac{\partial^2 G}{\partial \dot{x}\,\partial x} = 0$$

$$\frac{\partial G}{\partial \dot{x}} = 2t\dot{x} \qquad \frac{\partial^2 G}{\partial \dot{x}^2} = 2t > 0$$

$$\Delta_G = 0\cdot 2t - 0^2 = 0 \geq \underline{0}$$

Hence G is convex as a function of $(x,\dot{x})$.

The Euler equation :

$$0 = \frac{\partial G}{\partial x} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{x}}\right) = \underbrace{2t - \frac{d}{dt}(2t\dot{x})}$$

$$\left(= 2t - 2\dot{x} - 2t\ddot{x} \right.$$

or $\left. t\ddot{x} + \dot{x} = t \right)$

$$\frac{d}{dt}(t\dot{x}) = t$$

$$t\dot{x} = \frac{1}{2}t^2 + C_1$$

$$\dot{x} = \frac{1}{2}t + \frac{C_1}{t}$$

$$x(t) = \frac{1}{4}t^2 + C_1 \ln t + C_2$$

where

$$x(1) = \frac{1}{4} + C_1 \cdot 0 + C_2 = 0, \quad \underline{C_2 = \frac{-1}{4}}$$

$$x(2) = 1 + C_1 \ln 2 - \frac{1}{4} = 1$$

$$\underline{C_1 = \frac{1}{4\ln 2}}$$

Hence

$$x(t) = \frac{1}{4}t^2 + \frac{1}{4\ln 2}\ln t - \frac{1}{4} = \underline{\frac{1}{4}\left(t^2 + \frac{\ln t}{\ln 2} - 1\right)}$$

<u>solves the problem.</u>

<u>11. 2. 8</u>

11.2.2. $\min_{x} \int_0^1 (x^2 + tx + tx\dot{x} + \dot{x}^2)\,dt$, $x(0)=0$, $x(1)=$

Let

$$F(t,x,\dot{x}) = x^2 + tx + tx\dot{x} + \dot{x}^2$$

$$\frac{\partial F}{\partial x} = \underbrace{2x + t + t\dot{x}}, \quad \frac{\partial^2 F}{\partial x^2} = 2 > 0$$

$$\frac{\partial F}{\partial \dot{x}} = \underbrace{tx + 2\dot{x}}, \quad \frac{\partial^2 F}{\partial \dot{x}^2} = 2 > 0$$

$$\frac{\partial^2 F}{\partial \dot{x}\partial x} = t$$

$$\Delta_F = 2 \cdot 2 - t^2 = 4 - t^2 > 0$$

since $t \in [0,1]$

Hence F is (strictly) ~~strictly~~ convex as a function of $(x,\dot{x})$, for each $t \in [0,1]$

The Euler-equation:

$$0 = \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 2x + t + t\dot{x} - \frac{d}{dt}(tx + 2\dot{x})$$

$$= 2x + t + \underline{t\dot{x}} - x - \underline{t\dot{x}} - 2\ddot{x}$$

$$= t + x - 2\ddot{x}$$

or

$$\ddot{x} - \tfrac{1}{2}x = \frac{t}{2}$$

Char. equation : $r^2 - \frac{1}{2} = 0$, $r = \pm \frac{1}{\sqrt{2}}$

A complementary function is

$$x_c(t) = C_1 e^{\frac{1}{\sqrt{2}}t} + C_2 e^{-\frac{1}{\sqrt{2}}t}$$

We find a particular solution

$$x_p(t) = At + B$$

Then

$$\ddot{x}_p - \tfrac{1}{2}x_p = 0 - \tfrac{1}{2}At - \tfrac{1}{2}B \equiv \frac{t}{2}$$

$-\frac{1}{2}A = \frac{1}{2}$, $\underline{A = -1}$, $\underline{B = 0}$

Hence $\underline{x_p(t) = -t}$

$$x(t) = Ae^{t/\sqrt{2}} + Be^{-t/\sqrt{2}} - t$$

is a general solution.

Here

$x(0) = A + B = 0$, $B = -A$

$x(1) = A(e^{1/\sqrt{2}} - e^{-1/\sqrt{2}}) - 1 = 1$

$$x(t) = 2\frac{e^{t/\sqrt{2}} - e^{-t/\sqrt{2}}}{e^{1/\sqrt{2}} - e^{-1/\sqrt{2}}} - t$$ <u>solves the problem</u>

11.2.9

$$\min_x \int_0^1 (x^2 + t x \dot{x} + t^2 \dot{x}^2)\, dt$$

Euler-eq.:

$$0 = \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 2x + t\dot{x} - \frac{d}{dt}\left(tx + 2t^2 \dot{x}\right)$$

$$= 2x + t\dot{x} - x - t\dot{x} - 4t\dot{x} - 2t^2\ddot{x}$$

$$= x - 4t\dot{x} - 2t^2\ddot{x}$$

or $\quad 2t^2 \ddot{x} + 4t\dot{x} - x = 0$

Try $\quad x = t^r, \quad \dot{x} = r t^{r-1}, \quad \ddot{x} = r(r-1)t^{r-2}$

Then

$$2(r-1)r\, t^r + 4r\, t^r - t^r \equiv 0$$

$$2r^2 - 2r + 4r - 1 = 0 \qquad (\text{if } t \neq 0)$$

$$2r^2 + 2r - 1 = 0$$

$$r = \frac{1}{4}\left[-2 \pm \sqrt{4+8}\,\right] = \frac{1}{2}\left[-1 \pm \sqrt{3}\,\right]$$

$$r_1 = \frac{1}{2}(-1+\sqrt{3}), \quad r_2 = \frac{1}{2}(-1-\sqrt{3}) < 0$$

Hence t^{r_2} is not defined for $t = 0$.

$$x(t) = A\,t^{r_1} + B\,t^{r_2}$$

solves the equation, but is not C^2 on $[0,1]$.

EP

2.5 (10) A particular solution y_p of

$$y'' + 9y = 2\cos 3x + 3\sin 3x$$

The characteristic equation of $y'' + 9y = 0$ is $r^2 + 9 = 0$, $r = \pm 3i$. Hence $\cos 3x$ and $\sin 3x$ are solutions of the homogeneous equation. Hence we try

$$y_p(x) = x\,(c_1 \cos 3x + c_2 \sin 3x)$$

Then

$$y_p'(x) = c_1 \cos 3x + c_2 \sin 3x + x(-3c_1 \sin 3x + 3c_2 \cos 3x)$$

$$y_p''(x) = -3c_1 \sin 3x + 3c_2 \cos 3x + (-3c_1 \sin 3x + 3c_2 \cos 3x) + x(-9c_1 \cos 3x - 9c_2 \sin 3x)$$

and

$$y_p'' + 9y_p = -6c_1 \sin 3x + 6c_2 \cos 3x + \underline{x(-9c_1 \cos 3x - 9c_2 \sin 3x)} + \underline{x(9c_1 \cos 3x + 9c_2 \sin 3x)}$$

$$= -6c_1 \sin 3x + 6c_2 \cos 3x$$

$$\equiv 3 \sin 3x + 2 \cos 3x$$

so $-6c_1 = 3$, $c_1 = -\frac{1}{2}$; $6c_2 = 2$, $c_2 = \frac{1}{3}$

$$\underline{y_p(x) = x\left(-\tfrac{1}{2}\cos 3x + \tfrac{1}{3}\sin 3x\right)}$$

2.5 (61)

$x^2 y'' + xy' + y = \ln x$; $\boxed{y'' + \frac{1}{x} y' + \frac{1}{x^2} y = \frac{\ln x}{x^2}}$

$y_c = c_1 \cos(\ln x) + c_2 \sin(\ln x)$ is given.

We use Variation of Parameters ; $\boxed{y_c = u_1 \cos(\ln x) + u_2 \sin(\ln x)}$

This leads to (see EP) :

(1) $u_1' \cos(\ln x) + u_2' \sin(\ln x) = 0$ | $x > 0$

(2) $u_1' \left(-\frac{1}{x}\sin(\ln x)\right) + u_2' \frac{1}{x}\cos(\ln x) = \frac{\ln x}{x^2}$

From (1) $u_1' = -\tan(\ln x)\, u_2'$

Into (2) :

$$-\tan(\ln x)\cdot\left(-\frac{1}{x}\sin(\ln x)\right) u_2' + \frac{1}{x}\cos(\ln x)\, u_2' = \frac{\ln x}{x^2}$$

$$\frac{1}{x}\, \frac{\sin^2(\ln x) + \cos^2(\ln x)}{\cos(\ln x)}\, u_2' = \frac{\ln x}{x^2}$$

$$u_2(x) = \int u_2'\, dx = \int \underbrace{\frac{\ln x}{x} \cdot \cos(\ln x)\, dx}$$

$\left(v = \ln x \ , \quad dv = \frac{1}{x} dx \right)$

$$= \int \overset{f}{v}\, \overset{g'}{\cos v}\, dv = v \sin v - \int \sin v\, dv$$

$$= v \sin v + \cos v$$

(let integration constant equal to 0)

$u_2(x) = \ln x \sin(\ln x) + \cos(\ln x)$

$u_1'(x) = -\tan(\ln x) \cos(\ln x) \frac{\ln x}{x}$

$= -\sin(\ln x) \cdot \frac{\ln x}{x}$

$(v = \ln x)$

$u_1(x) = \dots = \ln x \cos(\ln x) - \sin(\ln x)$

Hence

$y_p(x) = u_1 \cos(\ln x) + u_2 \sin(\ln x)$

$= \ln x \cos^2(\ln x) - \underline{\sin(\ln x)\cos(\ln x)}$

$+ \ln x \sin^2(\ln x) + \underline{\sin(\ln x)\cos(\ln x)}$

$y_p(x) = \ln x \left(\cos^2(\ln x) + \sin^2(\ln x)\right) = \underline{\ln x}$

<u>SSS</u>:

<u>4.5</u> (2) $\quad f(x,y) = x^2 - xy + y^2 + 3x - 2y + 1$

is convex:

$\frac{\partial f}{\partial x} = 2x - y + 3, \quad \frac{\partial^2 f}{\partial x^2} = 2 > 0$

$\frac{\partial^2 f}{\partial y \partial x} = -1, \quad \frac{\partial f}{\partial y} = -x + 2y - 2, \quad \frac{\partial^2 f}{\partial y^2} = 2 > 0$

$$\Delta_f = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = 2 \cdot 2 - (-1)^2 = 3 > \underline{0}$$

Hence f is (strictly) convex in $\mathbb{R}^2$ by the 2nd-derivative test.

(3) $f(x,y) = x + y - e^x - e^{x+y}$

$\frac{\partial f}{\partial x} = 1 - e^x - e^{x+y}$, $\frac{\partial^2 f}{\partial x^2} = \underline{-e^x - e^{x+y}} < 0$

$\frac{\partial^2 f}{\partial y \partial x} = -e^{x+y}$,

$\frac{\partial f}{\partial y} = 1 - e^{x+y}$, $\underline{\frac{\partial^2 f}{\partial y^2} = -e^{x+y} < 0}$

$$\Delta_f = -(e^x + e^{x+y}) \cdot (-e^{x+y}) - (-e^{x+y})^2$$

$$= e^{x+y}(e^x + e^{x+y} - e^{x+y})$$

$$= e^{2x+y} > \underline{0}$$

Hence f is (strictly) concave in $\mathbb{R}^2$