Problem

$$F(x,y) = x^2(1+y^2)$$

(a) $R = \{(x,y) : |y| \leq \frac{1}{\sqrt{3}}\}$.

F is convex in R by the "2nd derivative test";

$$\frac{\partial F}{\partial x} = 2x(1+y^2) \quad , \quad \frac{\partial^2 F}{\partial x^2} = 2(1+y^2) > 0$$

$$\frac{\partial F}{\partial y} = 2x^2 y \qquad \frac{\partial^2 F}{\partial y^2} = 2x^2 \geq 0 \quad , \quad \frac{\partial^2 F}{\partial y \partial x} = 4xy$$

$$\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2 = 2(1+y^2)\cdot 2x^2 - (4xy)^2$$

$$= 4(x^2 + x^2y^2 - 4x^2y^2) = 4(x^2 - 3x^2y^2)$$

$$= 4x^2(1-3y^2) \geq 0 \iff 1-3y^2 \geq 0 \iff |y| \leq \frac{1}{\sqrt{3}}$$

Hence F is convex in R.

(b) $\min \int_0^1 x^2(1+\dot{x}^2)\,dt \quad , \quad x(0) = x(1) = 1$.

Euler-eq.:

$$0 = \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 2x(1+\dot{x}^2) - \frac{d}{dt}(2x^2\dot{x})$$

$$= 2x + 2x\dot{x}^2 - 4x\dot{x}^2 - 2x^2\ddot{x}$$

$$= 2x(1 - \dot{x}^2 - x\ddot{x}),$$

$$x\ddot{x} + \dot{x}^2 - 1 = 0 \quad \text{or} \quad x = 0.$$

(x could be 0 at some points t or over on some subintervals of [0,1])

(Problem)

The integrand $F(x,\dot{x}) = x^2(1+\dot{x}^2)$ does not contain t (explicitly). Hence we may consider the First-integral of the Euler-equation, that is:

$$F(x,\dot{x}) - \dot{x}\frac{\partial F}{\partial \dot{x}} = C,$$

$$x^2(1+\dot{x}^2) - \dot{x}(x^2 \cdot 2\dot{x}) = C,$$

$$x^2(1+\dot{x}^2 - 2\dot{x}^2) = C, \quad x^2(1-\dot{x}^2) = C$$

$$1-\dot{x}^2 = \frac{C}{x^2}, \quad \dot{x}^2 = 1 - \frac{C}{x^2},$$

$$\dot{x} = \pm\frac{\sqrt{x^2-C}}{|x|},$$

which is not particularly simple, but which can be solved by separation of variables:

$$\pm\int\frac{x\,dx}{\sqrt{x^2-C}} = \int dt.$$

(c) The only possible solution $x = x^*$ (of class C^2) is given by

$$\pm\sqrt{x^2-C} = t+k$$

$$x^2 - C = (t+k)^2$$

$x(0)=x(1)=1$ give $(1+k)^2 = k^2$, $\underline{k = -\frac{1}{2}}$. Then

$$1 - C = \frac{1}{4} \Rightarrow \underline{C = \frac{3}{4}}$$

(Problem)

The integrand $F(x,\dot{x}) = x^2(1+\dot{x}^2)$ does not contain t (explicitly). Hence we may consider the first-integral of the Euler-equation, that is:

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which is not particularly simple, but which can be solved by separation of variables:

$$\pm\int\frac{x\,dx}{\sqrt{x^2-C}} = \int dt$$

(b) The only possible solution $x = x^*(t)$ (of class C^2) is given by

$$\pm\sqrt{x^2-C} = t + k$$

$$x^2 - C = (t+k)^2$$

$x(0) = x(1) = 1$ give $(1+k)^2 = k^2$, $\underline{k = -\frac{1}{2}}$. Then

$$1 - C = \frac{1}{4} \Rightarrow \underline{C = \frac{3}{4}}$$

(Problem)

Hence

$$x^2 = \frac{3}{4} + \left(t - \frac{1}{2}\right)^2 = t^2 - t + 1$$

$$\underline{x(t) = \sqrt{t^2 - t + 1}},$$

where we must use a plus sign in front of the square root since $x(0) = 1 > 0$.

Next let (*)

$$V = \{x : x \in C^2[0,1],\ x(0) = x(1) = 1,\ |\dot{x}(t)| < \tfrac{1}{\sqrt{3}},\ \forall t \in [0,1]\}.$$

$x = x^* = \sqrt{t^2 - t + 1} \in V$:

We know that $x(0) = x(1) = 1$ and $x \in C^2[0,1]$.

It remains to show:

$$|\dot{x}(t)| < \frac{1}{\sqrt{3}}, \qquad t \in [0,1].$$

Now $\quad \dot{x}(t) = \dfrac{2t - 1}{2\sqrt{t^2 - t + 1}}$

Observe that

(*) Note that we use strict inequality, $|\dot{x}(t)| < \frac{1}{\sqrt{3}}$, in the definition of V, cfr. the argument at the end of the solution.

(Problem)

$$|\dot{x}(t)| < \frac{1}{\sqrt{3}}$$

$$\Updownarrow$$

$$\pm \dot{x}(t) < \frac{1}{\sqrt{3}}$$

$$\Updownarrow$$

$$\pm \sqrt{3}\,[2t-1] < 2\sqrt{t^2-t+1}$$

$$\Updownarrow$$

$$3(4t^2-4t+1) < 4(t^2-t+1)$$

$$\Updownarrow$$

$$g(t) = 8t^2 - 8t - 1 < 0$$

Now $g'(t) = 8(2t-1) = 0 \Leftrightarrow t = \frac{1}{2}$

which yields a minimum since $g''(t) = 16 > 0$.

Moreover, $g(0) = -1 < 0$, $g(1) = -1 < 0$

Hence $g(t) < 0$ for all $t \in [0,1]$.

Thus $x^* \in V$.

Since $F(x, \dot{x})$ is convex in $(x, \dot{x})$ for all $x \in V$ (by (a)) it follows that x^* minimizes the integral among all $x \in V$: The proof of the sufficient condition in the Calculus of Variation works in all open, convex subsets of the $x\dot{x}$-plane. In this case $\{(x, \dot{x}) : |\dot{x}| < \frac{1}{\sqrt{3}}\}$ is open and convex.

(Proof cont.)

$$|\dot{x}(t)| < \frac{1}{\sqrt{2}}$$

$$\Updownarrow$$

$$\pm \dot{x}(t) < \frac{1}{\sqrt{2}}$$

$$\Updownarrow$$

$$\pm\sqrt{3}\,(2t-1) < 2\sqrt{t^2-t+1}$$

$$\Updownarrow$$

$$3\,(4t^2-4t+1) < 4(t^2-t+1)$$

$$\Updownarrow$$

$$g(t) = 8t^2-8t-1 < 0$$

Now $g'(t) = 8(2t-1) = 0 \iff t = \frac{1}{2}$ which yields a minimum since $g''(t) = 16 > 0$.

Moreover, $g(0) = -1 < 0$, $g(1) = -1 < 0$.

Hence $g(t) < 0$ for all $t \in [0,1]$.

Thus $x^* \in V$.

Since $F(x,\dot{x})$ is convex in $(x,\dot{x})$ for all $x \in V$ (by (*1)) it follows that x^* minimizes the integral among all $x \in V$. The proof of the sufficient condition in the Calculus of Variations works in all open, convex subsets of the $x\dot{x}$-plane. In this case $\{(x,\dot{x}) : |\dot{x}| < \frac{1}{\sqrt{2}}\}$ is open and convex.

(Problem)

The proof of the sufficient condition requires the "Gradient Inequality" for convex (or concave) functions. This inequality requires the region to be open and convex.