Exam problems MAT 2500 with suggested solutions

PROBLEM 1

Show that the midpoints of the four sides of a quadrangle form a parallellogram whose area is half of the area of the original quadrangle. (hint: Show that each side of the new quadrangle is parallell to a diagonal of the old one.)

SOLUTION 1

Let A, B, C, D be the four vertices of the quadrangle with diagonals AC and BD. Let X be the midpoint of AB, Y the midpoint of BC, Z the midpoint of CD and W the midpoint of AD. The triangles XYB and ACB have a common angle, and the two sides AB and CB in the larger triangle are twice the length of the corresponding sides XB and YB in the smaller one. Therefore the two triangles are similar and XY is parallell to AC. Similarly ZW is parallell to AC, while YZ and XW are parallell to BD, so the quadrangle XYZW is a parallellogram. The area of the triangle ACB is four times the area of XYB. Similarly for the three other triangles. The area of XYZW is therefore:

$$(ACB - XYB) + (ACD - ZWD) - XWA - YZC =$$
$$= 3/4(ACB + ACD) - 1/4(BDA + BDC)$$
$$= 3/4ABCD - 1/4ABCD = 1/2ABCD$$

where ACB etc here means the area of the triangle ACB.

OPPGAVE 2

The eight points $(\pm 1, \pm 1, \pm 1)$ form the vertices of a cube. Four of these are vertices of a regular tetrahedron. The other four are vertices of another regular tetrahedron, Find the coordinates of the vertices of each tetrahedron. The intersection of the two tetrahedra form another polyhedron. Find the coordinates of the vertices of this new polyhedron. What kind of polyhedron is this?

SOLUTION 2

The four points

$$(1, 1, 1), (1, -1, -1), (-1, -1, 1), (-1, 1, -1)$$

and the four points

$$(1, 1, -1), (1, -1, 1), (-1, 1, 1), (-1, -1, -1)$$

are the vertices of two tetrahedra. Their edges are all diagonals of faces of the cube, so their intersection has vertices that are midpoints of the faces of the cube. Their coordinates are

$$(1, 0, 0), (-1, 0, 0), (0, -1, 0), (0, 1, 0), (0, 0, 1), (0, 0, -1)$$

and they are the vertices of a regular octahedron,

PROBLEM 3

The lines in the projective plane with equations $x + y = z$ and $x + y = 2z$ intersect the line $y = 0$ in A_1 and B_1 respectively, and the line $x = 0$ in A_2 and B_2 respectively. Let C be the intersection between $x = 0$ and $y = 0$, and let D be the point on the line $y = 0$ such that the crossratio $(A_1B_1, CD) = -1$. Find homogeneous coordinates for the point D. Let P be the intersection point between A_1B_2 and A_2B_1, and show that the lines A_1A_2, B_1B_2 and PD are concurrent on the line $z = 0$.

SOLUTION 3

In homogeneous coordinates we have:

$$A_1 = [1 : 0 : 1], B_1 = [2 : 0 : 1], A_2 = [0 : 1 : 1], B_2 = [0 : 2 : 1], C = [0 : 0 : 1]$$

Therefore $C = 2A_1 - B_1$ so if $D = xA_1 + yB_1$ and $(A_1B_1, CD) = -x/2y = -1$ we get $x = 2y$ and homogeneous coordinates (with y=1)

$$D = 2A_1 + B_1 = [4 : 0 : 3].$$

The equation of the line A_1B_2 is $2x + y - 2z = 0$, while the equation of the line A_2B_1 is $x + 2y - 2z = 0$. Their intersection point P has homogeneous coordinates $P = [2 : 2 : 3]$. The equation of the lines A_1A_2, B_1B_2 are $x + y - z = 0$ and $x + y - 2z = 0$ respectively. The intersection point of these lines has homogeneous coordinates $[1 : -1 : 0]$. The line DP has equation $3x + 3y - 4z = 0$ and passes also through the point $[1 : -1 : 0]$ on the line $z = 0$.

PROBLEM 4

Let L_1 be the line through the point $(1, 0)$ with slope k, and L_2 the line through the point $(-1, 0)$ with slope $1/k$. For which values of of k do the two lines intersect? Find the equation of the curve that the intersection point traces when k varies. What kind of curve is this?

SOLUTION 4

The equation of the line L_1 is $y = k(x-1)$, and the equation of the line L_2 is $y = 1/k(x+1)$. The two lines intersect when their slope are both defined and distinct, i.e. when $k \neq 0$, since the slope of L_2 is not defined at $k = 0$, and when $k \neq 1/k$, which means $k \neq \pm 1$. We get $k = y/(x-1)$ from L_1 so $y = (x-1)(x+1)/y$ from L_2. Therefore the intersection point between the two lines lies on the curve

$$x^2 - y^2 = 1$$

which defines a hyperbola.