UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Examination in:	MAT2700 — Introduction to mathematical finance and investment theory.
Day of examination:	Friday December 10, 2004.
Examination hours:	14.30 – 17.30.
This examination set consists of 4 pages.	
Appendices:	None.
Permitted aids:	None.

Make sure that your copy of the examination set is complete before you start solving the problems.

Problem 1.

For each of the two following market models analyse whether it admits arbitrage opportunities and whether it is complete.
State your conclusions clearly and prove them in full detail.

1. Single period market model:

   - Probability space $(\Omega, \mathcal{F}, P)$ with sample space $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and probability measure $P(\omega) > 0$, $\omega \in \Omega$.
   - Interest rate in the bank account: $r = 1/10$.
   - Two risky securities with prices given by

	$S_1(t,\omega)$		$S_2(t,\omega)$	
	$S_1(0,\omega)$	$S_1(1,\omega)$	$S_2(0,\omega)$	$S_2(1,\omega)$
ω_1	5	33/5	10	66/5
ω_2	5	33/5	10	44/5
ω_3	5	22/5	10	44/5

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2. Multi period binomial market model with one risky security the price of which is
$$S(t) = S(0)u^{N_t}d^{t-N_t}, \quad t = 1, ..., T,$$
where the size of an upward movement is $u = 2$, the size of a downward movement is $d = 1/2$ and the initial price is $S(0) = 5$. Here N_t, $t = 0, 1, ..., T$ ($N_0 \equiv 0$), is the stochastic process counting the number of upward movements occurring in the price.
Let $p = 3/4$ be the probability of the single upward movement.
(Remember that
$$P\{N_t = m\} = \binom{t}{m} p^m (1-p)^{t-m}, \qquad m = 0, 1, ..., T.)$$
Let the interest rate in the bank account be constant $r = 1/10$.

Problem 2.

1. Give the definition of *replicable contingent claim* in a multi period market model.

2. Let us consider the following multi period market model:

   - Probability space $(\Omega, \mathcal{F}, P)$ with sample space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and probability measure $P(\omega) > 0$, $\omega \in \Omega$.
   - Trading dates: $t = 0, 1, 2$
   - Interest rate in the bank account: $r \equiv 0$
   - One risky security with price process given by

$S(t,\omega)$	$S(0,\omega)$	$S(1,\omega)$	$S(2,\omega)$
ω_1	5	6	10
ω_2	5	6	4
ω_3	5	4	8
ω_4	5	4	2

   - The filtration $\mathbb{F}$ generated by the price process $S(t)$, $t = 0, 1, 2$. Namely, $\mathcal{F}_0$ trivial, $\mathcal{F}_1 = \{\emptyset, \{\omega : S(1,\omega) = 6\}, \{\omega : S(1,\omega) = 4\}, \Omega\}$ and $\mathcal{F}_2 = \mathcal{P}(\Omega)$.

   The unique risk neutral probability measure for this model is
$$Q(\omega) = \begin{cases} 1/6, & \omega = \omega_1, \\ 1/3, & \omega = \omega_2, \\ 1/6, & \omega = \omega_3, \\ 1/3, & \omega = \omega_4, \end{cases}$$
   (you are not required to prove this). With the above data,

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2a. find the replicating trading strategy and the price of the European call option

$$X_{call} = \big(S(2) - 5\big)^+;$$

2b. find the price of the European put option

$$X_{put} = \big(5 - S(2)\big)^+.$$

Problem 3.

Let $(\Omega, \mathcal{F}, Q)$ be a probability space equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t : t = 0, 1, ..., T\}$ where $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F} = \mathcal{P}(\Omega)$.

1. Give the definition of *martingale* with respect to the filtration $\mathbb{F}$ under the measure Q.

2. Let V_t^*, $t = 0, 1, ..., T$, be the discounted value process of a replicable contingent claim X, i.e.

$$V_t^* = E_Q\Big[\frac{X}{B_T}|\mathcal{F}_t\Big], \qquad t = 0, 1, ..., T,$$

(here B_t, $t = 0, 1, ..., T$, is the bank account process and Q is a risk neutral probability measure).
Prove that V_t^*, $t = 0, 1, ..., T$, is a martingale with respect to the filtration $\mathbb{F}$ under the measure Q.

Problem 4.

Let us consider the following complete single period market model:

- Probability space $(\Omega, \mathcal{F}, P)$ with sample space $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and probability measure

$$P(\omega) = \begin{cases} 1/3, & \omega = \omega_1, \\ 1/3, & \omega = \omega_2, \\ 1/3, & \omega = \omega_3, \end{cases}$$

- Interest rate in the bank account: $r = 0$

- Two risky securities with price processes given by

	$S_1(t,\omega)$		$S_2(t,\omega)$	
	$S_1(0,\omega)$	$S_1(1,\omega)$	$S_2(0,\omega)$	$S_2(1,\omega)$
ω_1	5	6	5	5
ω_2	5	8	5	3
ω_3	5	2	5	6

(Continued on page 4.)

The unique risk neutral probability measure in this model is:

$$Q(\omega) = \begin{cases} 1/2, & \omega = \omega_1, \\ 1/6, & \omega = \omega_2, \\ 1/3, & \omega = \omega_3, \end{cases}$$

(you are not required to prove this). With the above data, solve the following optimization problem

$$\begin{cases} \max_{H \in \mathbb{H}} E\big[u(V_1)\big] \\ V_0 = v \end{cases}$$

for the utility function

$$u(w) = \ln w, \qquad w > 0.$$

Here $\mathbb{H}$ is the set of all trading strategies and V_t, $t = 0, 1$, is the value process corresponding to the strategy $H \in \mathbb{H}$. Let $V_0 = v = 1$ be the initial value.

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