UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Examination in:	MAT2700 — Introduction to mathematical finance and investment theory.
Day of examination:	Wednesday, December 13, 2006.
Examination hours:	15.30 – 18.30.
This examination set consists of 3 pages.	
Appendices:	None.
Permitted aids:	None.

Make sure that your copy of the examination set is complete before you start solving the problems.

Problem 1.

Let us consider the following multi-period binomial market model with one risky security. Its price is given by

$$(*) \qquad S(t) = S(0)u^{N_t}d^{t-N_t}, \quad t = 1, ..., T,$$

where the initial price is $S(0) = 4$ and the parameters involved are fixed $u = 2$ and $d = 1/2$. Let $T = 2$. Here above N_t, $t = 0, 1, ..., T$ ($N_0 \equiv 0$), is the stochastic process counting the number of upward fluctuations occurring in the price. Remember that

$$P\{N_t = m\} = \binom{t}{m} p^m (1-p)^{t-m}, \qquad m = 0, 1, ..., T,$$

where $p \in (0, 1)$ is the probability of occurrence of a single upward movement.
In this market there is a non-risky form of investment possibility given by a bank account with interest rate r fixed at $r = 0$.

1. Draw a binomial tree diagram (or lattice) for the price values.

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2. Describe the *canonical* space $(\Omega, \mathcal{F}, P)$ for this market model.

3. Describe the filtration $\mathbb{F} = \{\mathcal{F}_t,\, t = 0, 1, 2\}$ generated by the price process $S(t)$, $t = 0, 1, 2$.

4. Find the risk neutral probability measure $Q(\omega)$, $\omega \in \Omega$.

5. Find the price of the following European contingent claims:

$$X_{\text{call}} = \big(S(2) - 5\big)^+ \qquad X_{\text{put}} = \big(5 - S(2)\big)^+$$

   at $t = 0$.

6. Verify that the discounted price process $S^*(t) = S(t)$, $t = 0, 1, ..., T$ $(r = 0)$, given in (*) is a martingale with respect to the filtration $\mathbb{F}$ and measure Q.

Problem 2.

Let us consider the following single period market model on the probability space $(\Omega, \mathcal{F}, P)$ with sample space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and probability measure $P(\omega) > 0$, $\omega \in \Omega$:

- A bank account with interest rate $r = 1/8$.
- Two risky securities with prices given by

	$S_1(0,\omega)$	$S_1(1,\omega)$	$S_2(0,\omega)$	$S_2(1,\omega)$
$\omega = \omega_1$	7	9	8	27/2
$\omega = \omega_2$	7	45/4	8	27/4
$\omega = \omega_3$	7	27/4	8	27/4
$\omega = \omega_4$	7	27/4	8	45/4

1. Prove that this market model has no arbitrage opportunities and that it is incomplete (i.e. find *all* risk neutral probability measures).

2. Characterize the set of all the replicable contingent claims.

3. Let

$$X_{\text{put}} = \big(e - S_1(1)\big)^+$$

   be an European put option with exercise price $e > 0$. Find for which values of e the claim X_{put} is replicable.

4. Let us consider the following two European call options

$$\hat{X}_{\text{call}} = \big(S_1(1) - 4\big)^+ \qquad \tilde{X}_{\text{call}} = \big(S_1(1) - 8\big)^+.$$

   Are they replicable? Find the interval of all non-arbitrage prices (i.e. fair prices) for these two claims.

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Problem 3.

Let us consider the following *complete* multi-period market model:

- Probability space $(\Omega, \mathcal{F}, P)$ with sample space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and probability measure

$$P(\omega) = \begin{cases} 5/14, & \omega = \omega_1, \\ 1/4, & \omega = \omega_2, \\ 1/4, & \omega = \omega_3, \\ 1/7, & \omega = \omega_4, \end{cases}$$

- A bank account with interest rate $r = 0$
- One risky security with price process given by

$S(t,\omega)$	$t = 0$	$t = 1$	$t = 2$
$\omega = \omega_1$	7	8	10
$\omega = \omega_2$	7	6	12
$\omega = \omega_3$	7	6	4
$\omega = \omega_4$	7	8	3

With the above data, find the optimal portfolio optimizing the problem

$$\begin{aligned} &\max_{H \in \mathbb{H}} E\big[u(V_2)\big] \\ &V_0 = v \end{aligned}$$

for the utility function

$$u(w) = \ln w, \qquad w > 0.$$

Here $\mathbb{H}$ is the set of all self-financing trading strategies and V_t, $t = 0, 1, 2$, is the value process corresponding a the strategy $H \in \mathbb{H}$. Let $v = 2$.
Apply either dynamic programming or martingale methods.

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