UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in MAT2700 — Mathematical Finance and Investment Theory.

Day of examination: Thursday, December 13th, 2012.

Examination hours: 14.30 – 18.30

This problem set consists of 6 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Løsningsforslag

Problem 1

a) To find the discounted process, we have to divide the values at time 1 by $B_1 = 1 + r = \frac{5}{4}$ and the values at time 2 by $B_2 = (1+r)^2 = \frac{25}{16}$. We get

	$\omega = \omega_1$	$\omega = \omega_2$	$\omega = \omega_3$	$\omega = \omega_4$
$S^*(\omega, 0)$	32	32	32	32
$S^*(\omega, 1)$	56	56	24	24
$S^*(\omega, 2)$	64	48	32	0

b) We look at all the underlying, one-period markets. In the first such market, the discounted process moves from 32 to 56 and 24, respectively, and hence we need to find a q such that $(56-32)q + (24-32)(1-q) = 0$. The solution is $q = \frac{1}{4}$, and hence $Q(\{\omega_1, \omega_2\}) = \frac{1}{4}$ and $Q(\{\omega_3, \omega_4\}) = \frac{3}{4}$. We analyze the other submarkets in the same way. In the market where the discounted process moves from 56 to 64 or 48, respectively, we get that both probabilities are $\frac{1}{2}$, and hence

$$Q(\omega_1) = Q(\omega_2) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$

In the market where the discounted process moves from 24 to 32 or 0, we get that the probabilities are $\frac{3}{4}$ and $\frac{1}{4}$, respectively, and hence

$$Q(\omega_3) = \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$$

$$Q(\omega_4) = \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16}$$

The martingale measure is unique as there are no other solutions of the equations we have to solve. This means that the market is abitrage free and

(Continued on page 2.)

complete.

c) Since the market is complete, the value of any claim is

$$V_0(X) = E_Q[X/B_2] = \frac{16}{25}E_Q[X]$$

In our case

$$V_0(X) = \frac{16}{25}\left(75 \cdot \frac{1}{8} + 50 \cdot \frac{1}{8} + 25 \cdot \frac{9}{16} + 0 \cdot \frac{3}{16}\right) = 6 + 4 + 9 + 0 = 19$$

d) We work backwards, solving the problem for one one-period submarket at a time. Look first at the submarket were the stock moves from 70 to 100 or 75. If H_0 is the investment in the bank account and H_1 the investment in the stock, we must have

$$\frac{25}{16}H_0 + 100H_1 = 75 \quad (1)$$
$$\frac{25}{16}H_0 + 75H_1 = 50 \quad (2)$$

If we solve this system, we get $H_0 = -16, H_1 = 1$. This means that

$$H_0(\omega_1, 2) = H_0(\omega_2, 2) = -16 \quad \text{and} \quad H_1(\omega_1, 2) = H_1(\omega_2, 2) = 1$$

We can now calculate the value of the option at time 1:

$$V_1(X)(\omega_1/\omega_2) = \frac{5}{4} \cdot (-16) + 1 \cdot 70 = 50$$

If we do the same for the submarket where the stock moves from 30 to 50 or 0, we get the equations

$$\frac{25}{16}H_0 + 50H_1 = 25 \quad (3)$$
$$\frac{25}{16}H_0 + 0 \cdot H_1 = 0 \quad (4)$$

If we solve this system, we get $H_0 = 0, H_1 = \frac{1}{2}$. This means that

$$H_0(\omega_3, 2) = H_0(\omega_4, 2) = 0 \quad \text{and} \quad H_1(\omega_3, 2) = H_1(\omega_4, 2) = \frac{1}{2}$$

The value of the option at time 1 is

$$V_1(X)(\omega_3/\omega_4) = \frac{5}{4} \cdot 0 + \frac{1}{2} \cdot 30 = 15$$

Finally, we look at the (first) submarket where the stock moves from 32 to 70 or 30. With the values we have computed for $V_1(X)$, we get the equations

$$\frac{5}{4}H_0 + 70H_1 = 50 \quad (5)$$
$$\frac{5}{4}H_0 + 30H_1 = 15 \quad (6)$$

(Continued on page 3.)

If we solve this system, we get $H_0 = -9, H_1 = \frac{7}{8}$. This means that

$$H_0(\omega) = -9 \quad \text{and} \quad H_1(\omega) = \frac{7}{8}$$

for all ω. The value of the portfolio at time 0 is

$$V_0(X) = -9 + 32 \cdot \frac{7}{8} = -9 + 28 = 19$$

in agreement with what we got in part c) above.

e) We first observe that $Y(\omega, t)$ is given by

	$\omega = \omega_1$	$\omega = \omega_2$	$\omega = \omega_3$	$\omega = \omega_4$
$Y(\omega, 0)$	3	3	3	3
$Y(\omega, 1)$	0	0	5	5
$Y(\omega, 2)$	0	0	0	35

To compute Z, we first observe that $Z(2) = Y(2)$. To find the value of $Z(t)$ for smaller times, we use the dynamic programming equation

$$Z_{t-1} = \max\{Y_{t-1}, E_Q[Z_t B_{t-1}/B_t \,|\, \mathcal{F}_{t-1}]\}$$

which in our setting becomes

$$Z_{t-1} = \max\{Y_{t-1}, E_Q[\frac{4}{5} Z_t \,|\, \mathcal{F}_{t-1}]\}$$

It is easy to see that $Z(\omega_1/\omega_2, 1) = 0$ (everything in the backward equation is 0). To compute $Z(\omega_3/\omega_4, 1) = 0$, we observe that

$$E_Q[\frac{4}{5} Z_t \,|\, \mathcal{F}_1] = \frac{3}{4} \cdot \frac{4}{5} \cdot 0 + \frac{1}{4} \cdot \frac{4}{5} \cdot 35 = 7$$

which is larger than $Y(\omega_3/\omega_4, 1) = 5$. Hence $Z(\omega_3/\omega_4, 1) = 7$. It remains to compute $Z(0)$. We have

$$E_Q[\frac{4}{5} Z_t \,|\, \mathcal{F}_0] = \frac{1}{4} \cdot \frac{4}{5} \cdot 0 + \frac{3}{4} \cdot \frac{4}{5} \cdot 7 = \frac{21}{5}$$

which is larger than $Y(0) = 3$. Hence $Z(0) = \frac{21}{5}$, and we have the following table:

	$\omega = \omega_1$	$\omega = \omega_2$	$\omega = \omega_3$	$\omega = \omega_4$
$Z(\omega, 0)$	$\frac{21}{5}$	$\frac{21}{5}$	$\frac{21}{5}$	$\frac{21}{5}$
$Z(\omega, 1)$	0	0	7	7
$Z(\omega, 2)$	0	0	0	35

As for stopping, it is not advantageous to exercise the option at $t = 0$ as $Z(0) > Y(0)$. At time $t = 1$, one should not stop on the lower (ω_3, ω_4)-branch of the tree since $Z(\omega_3/\omega_4, 1) > Y(\omega_3/\omega_4)$. On the upper (ω_1, ω_2)-branch it doesn't matter what we do as the value is always 0.

(Continued on page 4.)

Problem 2

a) The discounted process is

	$\omega = \omega_1$	$\omega = \omega_2$	$\omega = \omega_3$
$S_1^*(\omega, 0)$	9	9	9
$S_1^*(\omega, 1)$	12	8	4

and hence the risk neutral measures are given by the equations

$$3q_1 - q_2 - 5q_3 = 0$$
$$q_1 + q_2 + q_3 = 1$$

and the conditions $q_1, q_2, q_3 > 0$. If we solve the equations, we get

$$q_1 = \frac{1}{4} + q_3$$
$$q_2 = \frac{3}{4} - 2q_3$$
$$q_3 = q_3$$

where $0 < q_3 < \frac{3}{8}$ in order to satisfy the inequalities.

b) As a complete market has a unique risk neutral measure, $\mathcal{M}_1$ is not complete.

A contingent claim X is attainable if and only if $E_Q[X^*] = \frac{4}{5}E_Q[X]$ is the same for all risk neutral measures Q. Since

$$E_Q(X) = X(\omega_1)(\frac{1}{4} + q_3) + X(\omega_2)(\frac{3}{4} - 2q_3) + X(\omega_3)q_2 =$$
$$= \frac{1}{4}X(\omega_1) + \frac{3}{4}X(\omega_2) + q_3(X(\omega_1) - 2X(\omega_2) + X(\omega_3))$$

this means that X is attainable if and only if

$$X(\omega_1) - 2X(\omega_2) + X(\omega_3) = 0 \tag{7}$$

The claim $X(\omega_1) = 10$, $X(\omega_2) = 15$, $X(\omega_3) = 5$ does not satisfy this condition and is not attainable.

c) As we have just seen, $X = S_2(1)$ is not attainable. This means that $B_1, S_1(1), S_2(1)$ are linearly independent vectors in the three dimensional space of contingent claims over $\Omega = \{\omega_1, \omega_2, \omega_3\}$, and consequently the market is complete (see (1.22) in the textbook).

An alternative method is to show that $\mathcal{M}_2$ has a unique risk neutral measure. Since any risk neutral measure for $\mathcal{M}_2$ has to be a risk neutral measure for $\mathcal{M}_1$, it must be of the form described in part a). In addition we need to have

$$0 = E_Q[\Delta S_2^*(1)] = 0(\frac{1}{4} + q_3) + 4(\frac{3}{4} - 2q_3) + (-4)q_3 = 3 - 12q_3$$

We get $q_3 = \frac{1}{4}$, and thus the unique risk free measure for $\mathcal{M}_2$ is $Q(\omega_1) = \frac{1}{2}, Q(\omega_2) = \frac{1}{4}, Q(\omega_3) = \frac{1}{4}$.

(Continued on page 5.)

d) Since the market is complete, a consumption plan (C_0, C_1) is admissible if and only if $\nu = C_0 + E_Q[C_1/B_1]$. Hence we have to solve the constrained optimization problem:

MAXIMIZE: $\ln(C_0) + E[\ln(C_1)]$

SUBJECT TO: $300 = C_0 + \frac{4}{5}E_Q[C_1]$

If we use c_0, c_1, c_2, c_3 as names for $C_0, C_1(\omega_1), C_1(\omega_2), C_1(\omega_3)$, we may reformulate this as an ordinary Lagrange problem:

MAXIMIZE: $f(c_0, c_1, c_2, c_3) = \ln(c_0) + \frac{1}{3}\ln(c_1) + \frac{1}{3}\ln(c_2) + \frac{1}{3}\ln(c_3)$

SUBJECT TO: $300 = g(c_0, c_1, c_2, c_3) = c_0 + \frac{2}{5}c_1 + \frac{1}{5}c_2 + \frac{1}{5}c_3$

According to Lagrange's method, we are looking for points where $\nabla f = \lambda \nabla g$, i.e.

$$\frac{1}{c_0} = \lambda, \quad \frac{1}{3c_1} = \frac{2\lambda}{5}, \quad \frac{1}{3c_2} = \frac{\lambda}{5}, \quad \frac{1}{3c_3} = \frac{\lambda}{5}$$

Solving for the c_i's, we get

$$c_0 = \frac{1}{\lambda}, \quad , c_1 = \frac{5}{6\lambda}, \quad c_2 = \frac{5}{3\lambda}, \quad c_3 = \frac{5}{3\lambda}$$

If we substitute these expressions into the constraint, we get

$$300 = c_0 = \frac{1}{\lambda} + \frac{2}{5}\cdot\frac{5}{6\lambda} + \frac{1}{5}\cdot\frac{5}{3\lambda} + \frac{1}{5}\cdot\frac{5}{3\lambda} = \frac{2}{\lambda}$$

Hence $\lambda = \frac{1}{150}$, and we get $C_0 = c_0 = 150$, $C_1(\omega_1) = c_1 = 125$, $C_1(\omega_2) = c_2 = 250$, $C_1(\omega_3) = c_3 = 250$.

Problem 3

a) $V_-(X)$ and $V_+(X)$ are the lower and upper value of X at time 0, respectively. They can be described as

$$V_-(X) = \inf\{E_Q(X/B_1) \,|\, Q \text{ is a risk neutral measure}\} =$$
$$= \sup\{V_0(Y) \,|\, Y \text{ is an attainable claim }, Y \leq X\}$$

and

$$V_+(X) = \sup\{E_Q(X/B_1) \,|\, Q \text{ is a risk neutral measure}\} =$$
$$= \inf\{V_0(Y) \,|\, Y \text{ is an attainable claim }, Y \geq X\}$$

Since our claim in not attained, $E_Q[X/B_1]$ does not have the same value for all risk neutral Q, and hence $V_-(X) < V_+(X)$.

b) Assume for contradiction that the extended market is arbitrage free. Then there is a risk neutral measure Q for $\mathcal{M}^+$ such that $E_Q[X/B_1] = a$. This Q must also be a risk neutral measure for the old market $\mathcal{M}$, and hence $V_-(X) \leq E_Q[X/B_1] \leq V_+(X)$. This is a contradiction since a does not lie

(Continued on page 6.)

between $V_-(X)$ and $V_+(X)$.

c) To find a risk neutral measure for $\mathcal{M}^+$, it suffices to find a risk neutral measure Q for $\mathcal{M}$ such that $E_Q[X/B_1] = a$. Since $V_-(X) < a < V_+(X)$, there must be risk free measures Q_b and Q_c such that $E_{Q_b}[X/B_1] < a < E_{Q_c}[X/B_1]$. Let $b = E_{Q_b}[X/B_1]$ and $c = E_{Q_c}[X/B_1]$. Since $b < a < c$, there is a number λ strictly between 0 and 1 such that $a = \lambda b + (1-\lambda)c$ (in fact, $\lambda = \frac{c-a}{c-b}$). Since Q_b and Q_c are risk neutral measures for $\mathcal{M}$, so is the $Q = \lambda Q_b + (1-\lambda)Q_c$, and since

$$E_Q[X/B_1] = \lambda E_b[X/B_1] + (1-\lambda)E_c[X/B_1] = \lambda b + (1-\lambda)b = a$$

we have found a risk free measure for $\mathcal{M}^+$, and hence $\mathcal{M}^+$ is arbitrage free.