MAT2700 - MANDATORY ASSIGNMENT, FALL 2014

REMINDER: The assignment must be handed in before 15:00 on Thursday, October 30th, 2014, at the reception of the Department of Mathematics, in the 7th floor of Niels Henrik Abels hus, Blindern. Be careful to give reasons for your answers.

Exercise 1. Consider a financial market consisting of a sample space $\Omega = \{\omega_1, \omega_2, \omega_3\}$; a probability measure (vector) P with $P(\omega_1) = 1/6, P(\omega_2) = 1/6, P(\omega_3) = 2/3$; a bank account with $B_0 = 1$ and $B_1 = 1 + r$, where $r = 1/9$; and one risky asset, denoted by $S = \{S_t\}_{t=0,1}$,

$$S_0 = 3, \qquad S_1(\omega) = \begin{cases} 4, & \omega = \omega_1, \\ 3, & \omega = \omega_1, \\ 2, & \omega = \omega_3. \end{cases}$$

1a. Determine the risk-neutral probability measures.

What is the definition of an arbitrage opportunity? Is the market free of arbitrage?

1b. Is the market complete? Determine (characterize) the attainable claims.

1c. Determine the arbitrage-free prices of the claim $X = (4, 7/2, 4)$.

Exercise 2. Consider a market consisting of a bank account $B = \{B_0, B_1\}$ and one risky asset $S = \{S_0, S_1\}$. The current values of the assets are $B_0 = 1$ and $S_0 > 0$. The risk-free interest rate is $r > 0$. The sample space Ω consists of two states ω_1 and ω_2, where the probability of ω_1 occurring is $p \in (0, 1)$. The risky asset can either go up to $S_1 = uS_0$ (if $\omega = \omega_1$) or down to $S_1 = dS_0$ (if $\omega = \omega_2$), where $u > 1$ and $d < 1$ are given numbers.

2a. Determine the trading strategy that generates the claim X_u that gives 1 krone in the up state and nothing in the down state: $X_u = (1, 0)$. What is the initial portfolio value?

Similarly, determine the trading strategy and initial portfolio value of the claim X_d that gives 1 krone in the down state and nothing in the up state: $X_d = (0, 1)$.

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2b. Suppose $u > 1+r$, and then determine the risk-neutral probability measure Q.

2c. Take $S_0 = 100$, $u = 11/10$, $d = 9/10$, and $r = 1/20$. Use the risk-neutral valuation formula to find the prices of call and put options with exercise prices respectively 109 (call) and 91 (put).

Exercise 3. Consider the general market model given in Exercise 2, assuming $u > 1 + r$. Denote by $U(w)$ the utility function

$$U(w) = \ln(w), \qquad w > 0.$$

In what follows we are going to use the (two-steps) "risk-neutral probability" approach to solve the problem of maximizing expected utility of terminal wealth, with initial wealth $V_0 = \nu$, for a given $\nu > 0$:

$$\max_{H \in \mathbf{R}^2} E\left[U\left(V_1\right)\right], \qquad V_0 = \nu,$$

where $V_1 = H_0 B_1 + H_1 S_1$ is the terminal wealth.

3a. Using the Lagrange multiplier method, solve the problem of maximizing expected utility of wealth:

$$\max_{W \in \mathbb{R}^2} E[U(W)], \quad \text{subject to } E_Q[W/B_1] = \nu.$$

3b. Determine the optimal trading strategy H.