A ring
- abelian group
- multiplication is associative and distributive over add.
- commutative
- with identity element

$f: A \to B$ (- morphism preserves structure)

Motivating examples:
- $\mathbb{Z}$.
- Dedekind domains
  - Domain (no zero-divisors)
  - Noetherian
  - Dimension 1 (non-trivial primes are maximal)
  - Every non-zero proper ideal factors into primes.

E.g. $\mathbb{Z}[\omega]$ cyclotomic integers $\omega^p = 1$ ($x^p + y^p = z^p$ factors uniquely.)

- Polynomial rings $\mathbb{R}[x_1, \dots, x_n]$.

$I \triangleleft A$ ideal
- additive subgroup
- $A \cdot I \subseteq I$.

$\Rightarrow$ A/I ring, quotient ring or residueclass ring

Proposition

$I \triangleleft A$ fix ideal

$$\left\{ \underset{\text{ideal}}{I \subset J \triangleleft A} \right\} \overset{1-1}{\longleftrightarrow}_{\text{order-preserving}} \left\{ \underset{\text{ideal}}{\mathfrak{a} \triangleleft A/I} \right\}$$

$p: A \to A/I$

$$J \longmapsto \bar{J} = p(J)$$
$$p^{-1}(\mathfrak{a}) \longleftarrow \mathfrak{a}$$

Pf. We have $p^{-1}(\bar{J}) \supseteq J$ and $p^{-1}(\bar{J}) \subseteq J$ Why?

Further $p\, p^{-1}(\mathfrak{a}) \subseteq \mathfrak{a}$: $x \in p(p^{-1}(\mathfrak{a})) \Rightarrow \exists y \in p^{-1}(\mathfrak{a})$ s.t. $x = p(y)$
But then $p(y) \in \mathfrak{a}$, i.e. $x \in \mathfrak{a}$

and $p(p^{-1}(\mathfrak{a})) \supseteq \mathfrak{a}$: Let $x \in \mathfrak{a}$. Then by surjectivity $\exists y \in A$ s.t. $p(y) \in x$ and $y \in p^{-1}(\mathfrak{a})$ since $p(y) \in \mathfrak{a}$
$\Rightarrow x \in p(p^{-1}(\mathfrak{a}))$.

UMP:

[Universal Mapping Property]

$$\begin{tikzcd} P = R[x_1, \dots, x_n] \arrow[r, dashed] & R' \\ R \arrow[u, "\iota"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

+ map $\{x_1, \dots, x_n\} \to g(R')$ $\Rightarrow \exists \bar{\varphi}: P \to R'$

s. that $\bar{\varphi}\iota = \varphi$.

Proposition TFAE: i) A is a field

$A \neq 0$ ring.

ii) Only ideals in A are A and (0)

iii) Every $\varphi : A \to B (\neq 0)$ is injective,

Pf. i) $\Rightarrow$ ii) Let $\mathfrak{a} \neq (0)$. Then $\exists$ unit $x \in \mathfrak{a}$ (field prop)

$\Rightarrow A = (1) = (x) \subseteq \mathfrak{a} \Rightarrow \mathfrak{a} = A$

ii) $\Rightarrow$ iii) Suppose φ is not injective $\Rightarrow$ $\varphi^{-1}(0)$ non-trivial ideal in A.

$\Rightarrow \varphi^{-1}(0) = A$ since $\varphi^{-1}(0) \neq (0)$, i.e. $\varphi(A) = 0$

but $\varphi(1) = 1 \neq 0$. Contradiction.

iii) $\Rightarrow$ i) Let $x \in A$, $x \neq 0$. By assumption $\varphi : A \to A/(x)$ is injective

if $A/(x) \neq 0$. So $A/(x) = (0) \Rightarrow 1 \in (x)$ i.e. $\exists y \in A$

s.t. $xy = 1$. $\Rightarrow$ A is a field.

$R/\mathfrak{a}$ has UMP:

If $\pi(\mathfrak{a}) = 0$, given $\varphi : R \to R'$ s.t. $\varphi(\mathfrak{a}) = 0$, Then $\exists!$

map $\psi : R/\mathfrak{a} \to R'$ s.t. $\psi \pi = \varphi$.

$$\begin{tikzcd} R/\mathfrak{a} \arrow[r, dashed, "\psi"] & R' \\ R \supseteq \mathfrak{a} \arrow[u, "\pi"] \arrow[ur, "\varphi"'] & \end{tikzcd} \qquad \Rightarrow \exists \psi : R/\mathfrak{a} \to R'$$

Proposition

R ring. $\Rightarrow \ker \pi = (x-a)$

$P = R[X] \xrightarrow{\pi} R$

$x \longmapsto a$

evaluation map

Pf. • $y \in \ker \pi$, $y = f(x)$. Then $f(a) = 0$

$\Rightarrow (x-a) \mid f(x)$ i.e. $y = f(x) = (x-a)g(x) \in (x-a)$

• $\pi(g(x)(x-a)) = g(a)(a-a) = 0$.

$I, J \subset A$ ideals.

Sum $I + J$

Intersection $I \cap J$

Product $I \cdot J$ (generated by all products)

- commutative
- distributive $\mathfrak{a}\cdot(I+J) = \mathfrak{a}I + \mathfrak{a}J$

$$\mathfrak{a} \cap (I+J) = \mathfrak{a}\cap I + \mathfrak{a}\cap J \quad \text{if } \mathfrak{a} \supset I \text{ or } \mathfrak{a} \supset J$$

But $\mathfrak{a}\cap I + \mathfrak{a}\cap J \subseteq \mathfrak{a} \cap (I+J)$

$x \in \mathfrak{a}\cap I, y \in \mathfrak{a}\cap J \Rightarrow x+y \in \mathfrak{a}$ and $x+y \in I+J$.

Opposite: $\mathfrak{a} = (x), I = (x+y), J = (y) \subseteq k[x]$

$$\left.\begin{array}{l}(x) \cap (x+y, y) = (x) \cap (x,y) = (x) \\ (x)\cap(x+y) + (x)\cap(y) = (x^2+xy, xy) = (x^2, xy)\end{array}\right\} (x) \not\subseteq (x^2, xy)$$

- $(I+J)(I\cap J) \subseteq IJ$

$$(I+J)(I\cap J) = I\cdot(I\cap J) + J(I\cap J) \subseteq I\cdot J + JI = IJ.$$

Ex. $I = (x^2), J = (xy) \subseteq k[x,y]$

$$(I+J)(I\cap J) = \left((x^2)+(xy)\right)\left((x^2)\cap(xy)\right) = (x^2, xy)\cdot(x^2y) = (x^4y, x^3y^2)$$

$$I\cdot J = (x^2)(xy) = (x^3y) \not\subseteq (x^4y, x^3y^2).$$

$\Rightarrow IJ \subset I\cap J$ and $IJ = I \cap J$ iff $I+J = (1)$

Def I, J ideals coprime $\Leftrightarrow I+J = (1)$.

Prop

$\mathfrak{a}_i \subseteq A$ ideal $\quad A \xrightarrow{\varphi} \prod_{i=1}^{n} A/\mathfrak{a}_i$

$x \mapsto (x+\mathfrak{a}_1, \dots, x+\mathfrak{a}_n)$

i) $\mathfrak{a}_i, \mathfrak{a}_j$ coprime $\Rightarrow \prod \mathfrak{a}_i = \bigcap \mathfrak{a}_i$

ii) φ surjective $\Leftrightarrow \mathfrak{a}_i, \mathfrak{a}_j$ coprime $i \neq j$

iii) φ injective $\Leftrightarrow \bigcap \mathfrak{a}_i = (0)$.

Pf. i) For $n=2$, then $\mathfrak{a}_1 \cap \mathfrak{a}_2 = \mathfrak{a}_1 \cdot \mathfrak{a}_2$ if $\mathfrak{a}_1 + \mathfrak{a}_2 = (1)$

For $n>2$, assume true for $\mathfrak{a}_1, \ldots, \mathfrak{a}_{n-1}$. Let $\mathfrak{b} = \bigcap_{i=1}^{n-1} \mathfrak{a}_i = \prod_{i=1}^{n-1} \mathfrak{a}_i$

Then $\mathfrak{a}_n + \mathfrak{b} = (1)$. In fact for each i we have $\mathfrak{a}_n + \mathfrak{a}_i = (1)$

thus $\exists\, x_i \in \mathfrak{a}_i$, $y_i \in \mathfrak{a}_n$ s.t. $x_i + y_i = 1$

$\Rightarrow 1 - y_i = x_i \in \mathfrak{a}_i \Rightarrow \prod_{i=1}^{n-1} (1-y_i) \in \prod_{i=1} \mathfrak{a}_i = \mathfrak{b}$.

But $\prod (1-y_i) \in 1 + \mathfrak{a}_n$ i.e. $\mathfrak{b} + \mathfrak{a}_n = (1)$.

$\Rightarrow \mathfrak{b} \cap \mathfrak{a}_n = \mathfrak{b} \cdot \mathfrak{a}_n$ or $\bigcap \mathfrak{a}_i \cap \mathfrak{a}_n = \mathfrak{a}_n \cdot \prod \mathfrak{a}_i$.

ii) $\Rightarrow$ Let $(0, 1, 0) \in \prod A/\mathfrak{a}_i$. Then $\exists\, x \in A$ s.t. $x \in 1 + \mathfrak{a}_i$ and $x \in \mathfrak{a}_j$, $j \neq i$. But then $1 = (1-x) + x \in \mathfrak{a}_i + \mathfrak{a}_j$.

$\Leftarrow$ Fix i. Since $\mathfrak{a}_i + \mathfrak{a}_j = (1)$ we can find $x_j \in \mathfrak{a}_i$, $y_j \in \mathfrak{a}_j$ s.t. $x_j + y_j = 1$

Let $z = \prod_{j \neq i} y_j = \prod_{j \neq i} 1 - x_j \in 1 + \mathfrak{a}_i$. Then

$$\varphi(z) = \begin{cases} 0 & j \neq i \\ 1 & j = i \end{cases}$$

iii) $\ker \varphi = \bigcap_{i=1}^{n} \mathfrak{a}_i$.

$f: A \longrightarrow B$ ring homomorphism.

$\mathfrak{a} \subseteq A$ ideal $\quad \mathfrak{a}^e = B \cdot f(\mathfrak{a})$ extended

$\mathfrak{b} \subseteq B$ ideal $\quad \mathfrak{b}^c = f^{-1}(\mathfrak{b})$ contracted