$\mathfrak{p}$ prime ideal $\overset{\text{def}}{\iff}$ $R - \mathfrak{p}$ is multiplicatively closed and contains 1 $\iff$ $\mathfrak{p}$ proper and $xy \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$ or $y \in \mathfrak{p}$

$\Updownarrow$

$A/\mathfrak{p}$ domain (no zero-divisors)

$\underset{\text{domain}}{R} \Rightarrow \underset{\text{domain}}{R[X]}$ (and $R[X_1, \dots, X_n]$ domain)

$0 \neq p \in R$ prime element nonunit $\overset{\text{def}}{\iff}$ If $p \mid xy \Rightarrow p \mid x$ or $p \mid y$

$\Rightarrow$ $p$ prime iff $(p)$ prime

$0 \neq p \in R$ irreducible $\iff$ $p = yz \Rightarrow y$ or $z$ is a unit.

$R$ UFD : if any $x \in R$, $x \neq 0$ is a product of irreducible elements in a unique way up to order and units

1) In general : prime elements are irreducible

2) In a UFD: irreducible elements are prime

Pf 1) Suppose $p = yz$ and $p$ prime $\Rightarrow p \mid y$ or $p \mid z$ i.e $pa = y$ or $pb = z$

i.e $p = p\underbrace{az}_{=1}$ or $p = p\underbrace{by}_{=1}$

$\Rightarrow z$ or $y$ is a unit

2) Let $\underset{\text{irr.}}{p} \mid xy \Rightarrow x = p_1, \dots p_n$ $y = q_1 \dots q_m$

$p \mid xy = p_1 \cdots p_n q_1 \cdots q_m$

$\Rightarrow p = p_i$ ~~or~~ $p = q_j \Rightarrow p \mid x$ or $p \mid y$.

Notice

$f: A \to B$, $B \supseteq \mathfrak{p}$ prime $\Rightarrow f^{-1}(\mathfrak{p}) \subset A$ prime

$f$ surjective : $\underset{\text{prime}}{f^{-1}(\mathfrak{p}) \subseteq A} \Rightarrow \underset{\text{prime}}{\mathfrak{p} \subseteq B}$

Pf. $xy \in f^{-1}(\mathfrak{p}) \Rightarrow f(xy) = f(x) f(y) \in \mathfrak{p} \Rightarrow f(x) \in \mathfrak{p}$ or $f(y) \in \mathfrak{p}$

$\Rightarrow x \in f^{-1}(\mathfrak{p})$ or $y \in f^{-1}(\mathfrak{p})$.

$xy \in \mathfrak{p} \underset{\text{surjectivity}}{\Longrightarrow} \exists z, w \in A$ s.t. $f(z) = x,\ f(w) = y$

$\Rightarrow f(z) \cdot f(w) = f(zw) = xy$

$\Rightarrow z \cdot w = f^{-1}(xy) \in f^{-1}(\mathfrak{p})$

$\Rightarrow z \in f^{-1}(\mathfrak{p})$ or $w \in f^{-1}(\mathfrak{p})$

$f^{-1}(\mathfrak{p})$ prime $\Rightarrow x = f(z) \in \mathfrak{p}$ or $y = f(w) \in \mathfrak{p}$.

Definition

$\underline{m} \subset A$ maximal if $\underline{m}$ is proper with no proper ideal $\underline{m} \subsetneq \mathfrak{a} \subsetneq A$.

L. • $A$ is a field iff $(0)$ is a maximal ideal

• $\underline{m}$ is max ideal iff $A/\underline{m}$ is a field

Pf 1) Suppose $A$ is a field, $\mathfrak{a} \subset A$, $\mathfrak{a} \neq 0$. Then $\exists x \in \mathfrak{a}$, $x \neq 0$

$\Rightarrow x^{-1} x = 1 \in \mathfrak{a} \Rightarrow \mathfrak{a} = A$.

Suppose $(0)$ is max ideal. For $x \neq 0$ we have $(0) \subsetneq (x)$

thus $(x) = A$ i.e $\exists y \in A$ s.t. $xy = 1$.

2) $\underline{m}$ max $\Rightarrow$ $(0)$ max ideal in $A/\underline{m}$ and vice versa

( $A$: PID: Principle ideal domain: Every ideal is principal

Ex. $\mathbb{Z}$, $k[X]$ $k$: field

$p \in A$ irreducible $\Rightarrow$ $(p) \subset A$ maximal

Pf. $(p) \subsetneq (x) \Rightarrow p = xy$, $y$ nonunit

$\Rightarrow x$ unit $\Rightarrow (x) = A$.

$\Rightarrow A/(p)$ field )

<u>Theorem</u> Every proper ideal $\mathfrak{a}$ is contained in some max ideal

Pf. By Zorn's Lemma (or Axiom of choice).

$$S = \{\mathfrak{b} \subseteq A \, ; \, \mathfrak{a} \subseteq \mathfrak{b} \neq (1)\} \neq \emptyset \quad \text{since } \mathfrak{a} \in S$$

partially ordered by inclusion

$$\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq \mathfrak{b}_3 \subseteq \cdots\cdots \in S \, , \text{ Put } \mathfrak{b} = \bigcup_{i=1}^{\infty} \mathfrak{b}_i$$

Then.
- $\mathfrak{b}$ is an ideal (proper)
- $\mathfrak{a} \subseteq \mathfrak{b}$.
- $\mathfrak{b}$ is an upper bound for $\{\mathfrak{b}_i\}$ in $S$

$\underset{\text{Zorn's Lemma}}{\Longrightarrow}$ $S$ has a maximal element

$$\Longrightarrow \mathfrak{a} \subseteq \underset{\text{max. ideal}}{\underline{\mathfrak{m}}}$$

Corollary. $x \in A$. Then $\underset{\text{unit}}{x} \Leftrightarrow x \notin \underline{\mathfrak{m}}$ for all $\underset{\text{max}}{\underline{\mathfrak{m}}}$

Pf. $\underset{\text{unit}}{x} \Leftrightarrow (x) = A \Longrightarrow (x)$ is not contained in any max ideal $\underline{\mathfrak{m}}$

$\underset{\text{non-unit}}{x} \Rightarrow \underset{\text{proper ideal}}{(x) \subset A} \Longrightarrow (x) \subseteq \underset{\text{max}}{\underline{\mathfrak{m}}}$

Definition:

Jacobson radical $\quad rad(A) = \bigcap_{\text{all max}} \mathfrak{m}$

Nilradical $\quad N(A) = \bigcap_{\text{all primes}} \mathfrak{p} \quad = N(0)$

Radical of an ideal $\quad N(\mathfrak{a}) = \bigcap_{\substack{\mathfrak{a} \subseteq \mathfrak{p} \\ \text{prime}}} \mathfrak{p}$

Some propositions:

1. $x \in A$. Then $x \in rad(A) \Leftrightarrow \underset{\text{unit}}{1-xy} \; \forall y \in A$

Pf. $x \in rad(A)$, $\underset{\text{max}}{\mathfrak{m} \subset A}$. Suppose $1-xy \in \mathfrak{m}$. Then

$1-xy, x \in \mathfrak{m} \Rightarrow 1 \in \mathfrak{m}$. Contradiction.

$x \notin rad(A)$. Then $\exists \underset{\text{max}}{\mathfrak{m}}$ s.t. $x \notin \mathfrak{m}$

$\Rightarrow \mathfrak{m} + (x) = A$ i.e. $\exists m \in \mathfrak{m}, y \in A$ s.t. $m + xy = 1$

$\Rightarrow 1-xy = m \in \mathfrak{m}$, and $1-xy$ is non-unit.

2. $N(A) = \{x \in A ; \exists n \text{ s.t. } x^n = 0\}$ (nilpotent elements)

Pf. $\supseteq$: If $x^n = 0 \in \mathfrak{p}$, then $x \in \mathfrak{p} \quad \forall \mathfrak{p} \Rightarrow x \in N(A)$

$\subseteq$: $x \in N(A) \Rightarrow x \in \mathfrak{p} \; \forall$ prime $\mathfrak{p}$ and suppose $x$ is not nilpotent

$$\Sigma = \{\mathfrak{a} \subseteq A \mid x^n \notin \mathfrak{a} \text{ for all } n > 0\}$$

Nonempty: $(0) \in \Sigma$

Ordered by inclusion

$\Rightarrow \Sigma$ has a maximal element: $\mathfrak{p}$

Let $y, z \notin \mathfrak{p} \Rightarrow \mathfrak{p} + (z), \mathfrak{p} + (y) \notin \Sigma$ (strictly bigger)

$\Rightarrow x^n \in \mathfrak{p} + (z), x^m \in \mathfrak{p} + (y) \Rightarrow x^{n+m} \in \mathfrak{p} + (zy)$

$\Rightarrow \mathfrak{p} + (zy) \notin \Sigma$, thus $\mathfrak{p} + (zy) \notin \Sigma \Rightarrow zy \notin \mathfrak{p}$

$\Rightarrow \mathfrak{p}$ prime.

Maximal element in $\Sigma$ is a prime ideal $\mathfrak{p}$. and $x^n \notin \mathfrak{p}$ for any $n>0$. $\Rightarrow$ $x \notin \mathfrak{p}$ and $x \notin N(A)$ Contradiction

$\Rightarrow$ Thus $x$ must be nilpotent.

Ex

$$N((8)) = (2)$$
$$N((12)) = (6) = (2) \cap (3)$$

$$N((x^n)) = (x).$$