

Definition

A sequence

$$\rightarrow M_{i-1} \xrightarrow{d_{i-1}} M_i \xrightarrow{d_i} M_{i+1} \rightarrow \dots$$

is said to be exact at M_i if $\text{Ker}(d_i) = \text{Im}(d_{i-1})$.

The sequence is exact if it is exact at all M_i .

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \quad \text{short-exact}$$

Ex 1) $0 \rightarrow M \xrightarrow{\oplus} M \oplus N \rightarrow N \rightarrow 0$

2) $0 \rightarrow M' \rightarrow M \rightarrow M/M' \rightarrow 0$

Notice: exactness commutes with direct product and direct sums.

Notice: M' and M'' are fin. generated $\Rightarrow M$ finitely generated

Ex 3) $0 \rightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \rightarrow \mathbb{Z}/(n) \rightarrow 0$ is exact

but no sequence $\mathbb{Z}/(n) \rightarrow \mathbb{Z} \rightarrow \dots$

Definition:

The SES $0 \rightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \rightarrow 0$ splits if

\exists isomorphism $\varphi: M \rightarrow M' \oplus M''$ such that $\varphi\alpha = \text{id}_{M'}$

and $\beta\varphi = \text{id}_{M''}$

equivalently

$$\begin{array}{ccccccc}
 0 & \rightarrow & M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' \rightarrow 0 \\
 & & & & \downarrow \varphi & & \uparrow \text{pr}_2 \\
 & & & & M' \oplus M'' & &
 \end{array}$$

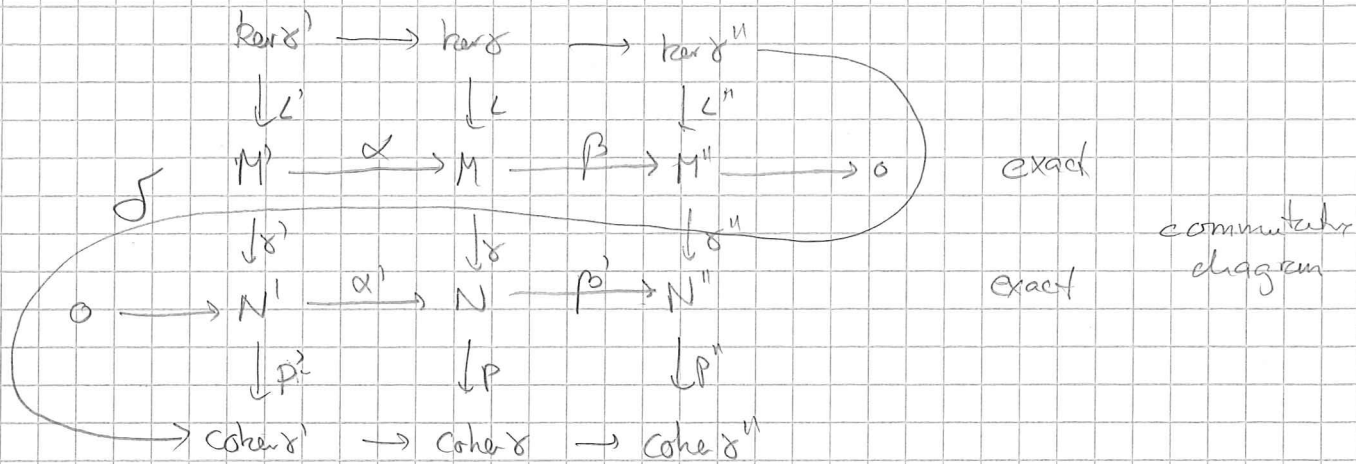
$$\begin{array}{ccc}
 M' & \xleftarrow{\sigma} & M \\
 & \alpha & \downarrow \\
 & & M' \oplus M''
 \end{array}$$

σ retraction of α if $\sigma\alpha = \text{id}_{M'}$

$$\begin{array}{ccc}
 M & \xleftarrow{\sigma} & M \\
 & \beta & \downarrow \\
 & & M''
 \end{array}$$

σ section of β if $\beta\sigma = \text{id}_{M''}$

Snake Lemma



Pf By diagram chasing.

Ex Define snake: Let $x \in \ker \alpha''$. Then $\exists y \in M$ s.t. $\beta(y) = \mathcal{L}''(x)$

Consider $\gamma(y)$. We have $\beta' \gamma(y) = \gamma'' \beta(y) = \gamma''(\mathcal{L}''(x)) = 0$. Thus

$\gamma(y) \in \ker \beta' = \text{im } \alpha'$, i.e. $\gamma(y) = \alpha'(z)$ for some $z \in N'$

Put $\mathcal{J}(x) = \mathcal{P}'(z)$. Well-defined since another choice of

y , i.e. y' s.t. $\beta(y') = \mathcal{L}''(x)$, gives $\beta(y-y') = \mathcal{L}''(x) - \mathcal{L}''(x) = 0$

Thus $\exists w \in M'$ s.t. $\alpha(w) = y-y'$. But then:

$$\alpha' \gamma(w) = \gamma' \alpha(w) = \gamma'(y-y') = \gamma'(y) - \gamma'(y')$$

But $\gamma'(y) = \alpha'(z)$ and $\gamma'(y') = \alpha'(z')$ (thus $y-y'$)

$$\alpha'(\gamma(w)) = \alpha'(z) - \alpha'(z') = \alpha'(z-z')$$

Since α' is injective we get $\gamma(w) = z-z'$ and

$$\mathcal{P}'(z-z') = \mathcal{P}' \gamma(w) = 0.$$

Hom is left exact.:

$$1) \quad M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \rightarrow 0 \iff 0 \rightarrow \text{Hom}(M'', N) \xrightarrow{\beta^*} \text{Hom}(M, N) \xrightarrow{\alpha^*} \text{Hom}(M', N)$$

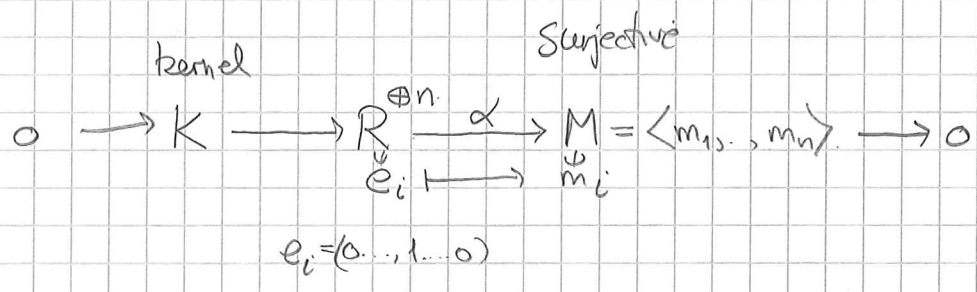
exact exact

$$2) \quad 0 \rightarrow N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \iff 0 \rightarrow \text{Hom}(M, N) \xrightarrow{\alpha_*} \text{Hom}(M, N) \xrightarrow{\beta_*} \text{Hom}(M, N'')$$

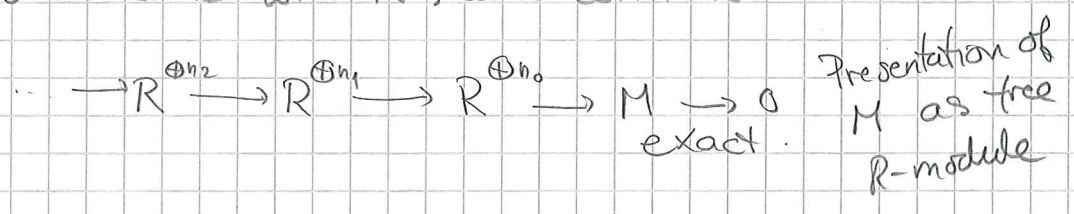
exact exact

Pf. 2) • Let $\varphi: M \rightarrow N'$ s.t. $\alpha_* \varphi = \alpha \varphi = 0$. Then $\varphi(x) = 0 \forall x$ and $\varphi = 0$.

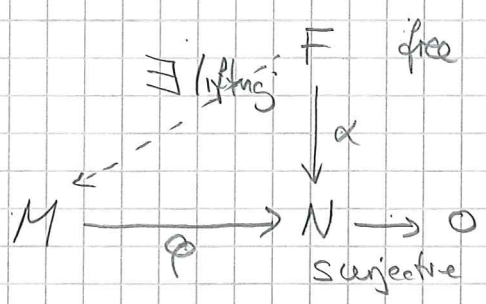
! since α is injective
 • Let $\psi: M \rightarrow N$ s.t. $\beta_* \psi = \beta \psi = 0$. Then $\psi(x) \in \ker \beta = \text{im } \alpha$ and $\exists! y \in N'$ s.t. $\psi(x) = \alpha(y)$. Define $\varphi: M \rightarrow N'$ by $\varphi(x) = y$. Then $\psi = \alpha_* \varphi$ ($\psi(x) = \alpha(y) = \alpha \varphi(x) = \alpha_* \varphi(x)$)



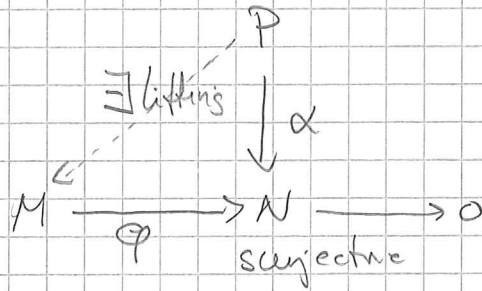
Do the same with K , and continue



Property of free module



Projective module

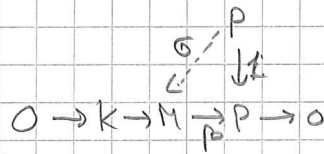


Thm

TFAE:

- 1) P projective
- 2) $0 \rightarrow K \rightarrow M \rightarrow P \rightarrow 0$ splits
- 3) $\exists K$ s.t. $K \oplus P$ free
- 4) $N' \rightarrow N \rightarrow N'' \Rightarrow \text{Hom}(P, N') \rightarrow \text{Hom}(P, N) \rightarrow \text{Hom}(P, N'')$
exact exact
- 5) $M \xrightarrow{\beta} N \Rightarrow \text{Hom}(P, M) \xrightarrow{\beta^*} \text{Hom}(P, N)$
surjective surjective

Pf. 1) \Rightarrow 2)



$$2) \Rightarrow 3) \quad \text{Presentation } 0 \rightarrow K \rightarrow R^{\oplus n} \rightarrow P \rightarrow 0 \Rightarrow R^{\oplus n} \simeq K \oplus P.$$

$$3) \Rightarrow 4) \quad \text{Hom}(R^{\oplus n}, N) \simeq \bigoplus \text{Hom}_R(R, N) \simeq \bigoplus N.$$

$$\text{Hom}(K \oplus P, N) \simeq \text{Hom}(K, N) \oplus \text{Hom}(P, N).$$

$$4) \Rightarrow 5) \quad \text{Put } N'' = 0$$

$$5) \Rightarrow 1) \quad \text{By definition.}$$

Some examples:

- 1) $A \supseteq \underline{m}$ ^{max ideal}
local ring. Suppose $\{x_1, \dots, x_n\}$ generate $F = A/\underline{m} \cong A^n$. Then $\{x_1, \dots, x_n\}$ is a basis for F .

Pf. Define

$$0 \rightarrow K \rightarrow A^n \xrightarrow{\varphi} A^n \rightarrow 0$$

\parallel $x_i \mapsto e_i$
 $\ker \varphi$

Dividing out by max ideal gives

$$0 \rightarrow K/\underline{m}K \rightarrow k^n \xrightarrow{\text{iso}} k^n \rightarrow 0$$

$\Rightarrow K/\underline{m}K = 0$

K fin. gen., $\underline{m} \subset (A)$, $K = \underline{m}K$

Nakayama $\Rightarrow K = 0$.

2) $0 \rightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \rightarrow \mathbb{Z}/(n) \rightarrow 0$

Apply $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/(m))$, notice $\text{Hom}(\mathbb{Z}, \mathbb{Z}/(m)) \cong \mathbb{Z}/(m)$

$$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Z}/(m)) \rightarrow \mathbb{Z}/(m) \xrightarrow{\cdot n} \mathbb{Z}/(m) \rightarrow ?$$

\parallel
 \mathbb{Z}

$$\varphi: \mathbb{Z}/(m) \rightarrow \mathbb{Z}/(m)$$

$$1 \mapsto a$$

$$0 = n \cdot 1 \mapsto n \cdot a = 0 \quad \text{i.e. } na \in (m) \quad \text{i.e. } a \in (m;n)$$

$$\Rightarrow a \in \left(\frac{m}{\gcd(m;n)} \right)$$

Ex $n=12$ $m=15$ $12 \cdot a \in (15) \Rightarrow a \in (5) = \left(\frac{15}{3} \right)$

Continue

$$\Rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n), \mathbb{Z}/(m)) \cong \mathbb{Z}/(\gcd(n,m))$$

$$\text{coker}(\mathbb{Z}/(m) \xrightarrow{\cdot n} \mathbb{Z}/(m)) \cong \mathbb{Z}/(m) / \gcd(n,m)\mathbb{Z}/(m) \cong \mathbb{Z}/(\gcd(n,m))$$

$$0 \rightarrow \mathbb{Z}/(3) \xrightarrow{\cdot 5} \mathbb{Z}/(15) \xrightarrow{\cdot 12} \mathbb{Z}/(15) \rightarrow \mathbb{Z}/(3) \rightarrow 0$$