Definition A sequence

$$\to M_{i-1} \xrightarrow{d_{i-1}} M_i \xrightarrow{d_i} M_{i+1} \to \cdots$$

is said to be <u>exact at M_i</u> if $\operatorname{Ker}(d_i) = \operatorname{Im}(d_{i-1})$.

The sequence is <u>exact</u> if it is exact at all M_i.

$$0 \to M' \to M \to M'' \to 0 \qquad \text{short exact}$$

Ex. 1) $0 \to M \to M \oplus N \to N \to 0$

2) $0 \to M' \to M \to M/M' \to 0$

Notice: exactness commutes with direct product and direct sums.

Notice: M' and M'' are fin. generated $\Rightarrow$ M finitely generated

Ex 3) $0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/(n) \to 0$ is exact.

but no sequence $\mathbb{Z}/(n) \to \mathbb{Z} \to \cdots$

Definition: The SES $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ splits if $\exists$ isomorphism $\varphi : M \to M' \oplus M''$ such that $\varphi\alpha = id_M$ and $\beta = pr_{M''}\varphi$

equivalently

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "\alpha"] \arrow[dr, "id_{M'}"'] & M \arrow[r, "\beta"] \arrow[d, "\varphi"] & M'' \arrow[r] & 0 \\
 & & M' \oplus M'' \arrow[ur, "pr_2"'] & &
\end{tikzcd}$$

$$\begin{tikzcd} M' \arrow[r, "\alpha"'] & M \arrow[l, bend right, "\rho"'] \end{tikzcd}$$ ρ retraction of α if $\rho\alpha = id_{M'}$

$$\begin{tikzcd} M \arrow[r, "\beta"'] & M'' \arrow[l, bend right, "\sigma"'] \end{tikzcd}$$ σ section of β if $\beta\sigma = id_{M''}$

<u>Proposition</u>

$$0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0 \qquad \text{SES}$$

TFAE:
1) The sequence splits
2) $\exists$ retraction $\rho : M \to M'$ of α
3) $\exists$ section $\sigma : M'' \to M$ of β

<u>Pf</u> 1) $\Rightarrow$ 2), 3)

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "\alpha"] \arrow[dr, "\iota_1"'] & M \arrow[r, "\beta"] \arrow[d, "\varphi"] & M'' \arrow[r] & 0 \\
& & M' \oplus M'' \arrow[ur, "p_2"'] & &
\end{tikzcd}$$

Retraction $p_1 \varphi$, section $\varphi^{-1}\iota$

3) $\Rightarrow$ 1),

$$\begin{tikzcd}
M \arrow[r, shift right, "\beta"'] & M'' \arrow[l, shift right, "\sigma"']
\end{tikzcd}$$

Put $\rho' = 1_M - \sigma\beta$. Then $\beta\rho' = \beta - \beta(\sigma\beta) = \beta - (\beta\sigma)\beta = \beta - \beta = 0$

$\Rightarrow \rho'(M) \subset \ker(\beta) = \operatorname{im}\alpha$. Thus $\forall m \in M$, $\exists! m' \in M'$ s.t.

$\rho'(m) = \alpha(m')$, i.e. $m - \sigma\beta(m) = \alpha(m')$

or $m = \alpha(m') + \sigma\beta(m)$. Define $\varphi : M \to M' \oplus M''$

by $\varphi(m) = (m', \beta(m))$. This is a module homomorphism

It is injective since $m' = \beta(m) = 0 \Rightarrow m = 0$ and surjective

since if $(m', m'') \in M' \oplus M''$, then $(m', m'') = \varphi(\alpha(m') + \sigma(m''))$

Why? $\rho'(\alpha(m') + \sigma(m'')) = \alpha(m') + \sigma(m'') - \sigma\beta(\alpha(m') + \sigma(m''))$

$$= \alpha(m') + \sigma(m'') - 0 - \sigma \underbrace{\beta\sigma}_{=\mathrm{id}}(m'')$$

$$= \alpha(m') + \sigma(m'') - \sigma(m'') = \alpha(m')$$

Thus $\varphi(\alpha(m') + \sigma(m'')) = (m', \beta(\alpha(m') + \sigma(m''))) = (m', m'')$.

Snake Lemma

$$
\begin{tikzcd}
 & \ker\gamma' \arrow[r] \arrow[d, "\iota'"] & \ker\gamma \arrow[r] \arrow[d, "\iota"] & \ker\gamma'' \arrow[d, "\iota''"] \arrow[ddll, bend left, "\delta"'] & \\
 & M' \arrow[r, "\alpha"] \arrow[d, "\gamma'"] & M \arrow[r, "\beta"] \arrow[d, "\gamma"] & M'' \arrow[r] \arrow[d, "\gamma''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "\alpha'"] \arrow[d, "p'"] & N \arrow[r, "\beta'"] \arrow[d, "p"] & N'' \arrow[d, "p''"] & \\
 & \operatorname{coker}\gamma' \arrow[r] & \operatorname{coker}\gamma \arrow[r] & \operatorname{coker}\gamma'' &
\end{tikzcd}
$$

exact

exact

commutative diagram

Pf By diagram chasing.

Ex Define snake: Let $x \in \ker\gamma''$. Then $\exists\, y \in M$ s.t. $\beta(y) = \iota''(x)$.

Consider $\gamma(y)$. We have $\beta'\gamma(y) = \gamma''\beta(y) = \gamma''(\iota''(x)) = 0$. Thus $\gamma(y) \in \ker\beta' = \operatorname{im}\alpha'$, i.e. $\gamma(y) = \alpha'(z)$ for some $z \in N'$.

Put $\delta(x) = p'(z)$. Well-define since another choice of y, i.e. y' s.t. $\beta(y') = \iota''(x)$, gives $\beta(y-y') = \iota''(x) - \iota''(x) = 0$.

Thus $\exists\, \omega \in M'$ s.t. $\alpha(\omega) = y - y'$. But then

$\alpha'\gamma'(\omega) = \gamma\alpha(\omega) = \gamma(y-y') = \gamma(y) - \gamma(y')$

But $\gamma(y) = \alpha'(z)$ and $\gamma(y') = \alpha'(z')$. Thus

$$\alpha'(\gamma'(\omega)) = \alpha'(z) - \alpha'(z') = \alpha'(z-z')$$

Since α' is injective we get $\gamma'(\omega) = z - z'$ and

$$p'(z-z') = p'\gamma'(\omega) = 0.$$

Hom is left exact:

1) $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ exact $\iff$ $0 \to \mathrm{Hom}(M'',N) \xrightarrow{\beta^*} \mathrm{Hom}(M,N) \xrightarrow{\alpha^*} \mathrm{Hom}(M',N)$ exact

2) $0 \to N' \xrightarrow{\alpha} N \xrightarrow{\beta} N''$ exact $\iff$ $0 \to \mathrm{Hom}(M,N') \xrightarrow{\alpha_*} \mathrm{Hom}(M,N) \xrightarrow{\beta_*} \mathrm{Hom}(M,N'')$ exact.

Pf. 2) • Let $\varphi : M \to N'$ s.t. $\alpha_* \varphi = \alpha\varphi = 0$. Then $\varphi(x) = 0$ $\forall x$ and $\varphi = 0$.

• Let $\psi : M \to N$ s.t. $\beta_* \psi = \beta\psi = 0$. Then $\psi(x) \in \ker\beta = \operatorname{im}\alpha$ (for $x \in M$) and $\exists! y \in N'$ s.t. $\psi(x) = \alpha(y)$. Define $\varphi : M \to N'$ by $\varphi(x) = y$. Then $\psi = \alpha_* \varphi$ $\quad(\psi(x) = \alpha(y) = \alpha\varphi(x) = \alpha_*\varphi(x))$

! since α is injective

$$\begin{array}{ccccccccc} & & \text{kernel} & & & \text{surjective} & & & \\ 0 & \longrightarrow & K & \longrightarrow & R^{\oplus n} & \xrightarrow{\alpha} & M = \langle m_1, \dots, m_n \rangle & \longrightarrow & 0 \\ & & & & e_i & \longmapsto & m_i & & \end{array}$$

$e_i = (0 \dots, 1 \dots 0)$

Do the same with K, and continue

$$\cdots \longrightarrow R^{\oplus n_2} \longrightarrow R^{\oplus n_1} \longrightarrow R^{\oplus n_0} \longrightarrow M \longrightarrow 0 \quad \text{exact.}$$

Presentation of M as free R-module

Property of free module

$$\begin{tikzcd} & F \arrow[dl, dashed, "\exists \text{ lifting}"'] \arrow[d, "\alpha"] & \text{free} \\ M \arrow[r, "\varphi"'] & N \arrow[r] & 0 \end{tikzcd}$$

surjective

Projective module

$$
\begin{tikzcd}
 & P \arrow[d, "\alpha"] \arrow[dl, dashed, "\exists\ \text{lifting}"'] & \\
M \arrow[r, "\varphi"'] & N \arrow[r] & 0
\end{tikzcd}
$$

surjective

Thm

TFAE:

1) P projective
2) $0 \to K \to M \to P \to 0$ splits
3) $\exists\, K$ s.t. $K \oplus P$ free
4) $\underset{\text{exact}}{N' \to N \to N''} \Rightarrow \underset{\text{exact}}{Hom(P,N'') \to Hom(P,N) \to Hom(P,N')}$
5) $\underset{\text{surjective}}{M \overset{\beta}{\twoheadrightarrow} N} \Rightarrow \underset{\text{surjective}}{Hom(P,M) \overset{\beta^*}{\to} Hom(P,N)}$

Pf. 1) ⇒ 2)

$$
\begin{tikzcd}
 & & & P \arrow[d, "1"] \arrow[dl, dashed, "\sigma"'] & \\
0 \arrow[r] & K \arrow[r] & M \arrow[r, "p"'] & P \arrow[r] & 0
\end{tikzcd}
$$

2) ⇒ 3) Presentation $0 \to K \to R^{\oplus n} \to P \to 0 \Rightarrow R^{\oplus n} \simeq K \oplus P$.

3) ⇒ 4) $Hom(R^{\oplus n}, N) \simeq \oplus Hom_R(R,N) \simeq \oplus N$.

$Hom(K \oplus P, N) \simeq Hom(K,N) \oplus Hom(P,N)$.

4) ⇒ 5) Put $N'' = 0$

5) ⇒ 1) By definition.

Some examples:

1) $A \supseteq \underline{m}$ max ideal, local ring. Suppose $\{x_1, \dots, x_n\}$ generate $F = A^{\oplus n} = A^n$. Then $\{x_1, \dots, x_n\}$ is a basis for F.

Pf. Define

$$0 \to \underset{\ker\varphi}{K} \to A^n \xrightarrow{\varphi} A^n \to 0$$
$$x_i \longmapsto e_i$$

Dividing out by max ideal gives

$$0 \to K/\underline{m}\cdot K \to k^n \xrightarrow[\text{iso}]{} k^n \to 0$$

$$\Rightarrow K/\underline{m}K = 0.$$

$\underset{\text{fin. gen}}{K}$, $\underline{m} \subset r(A)$, $K = \underline{m}K$

Nakayama $\Rightarrow K = 0$.

2)

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/(n) \to 0$$

Apply $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/(m))$, notice $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/(m)) \approx \mathbb{Z}/(\underline{m})$

$$\underset{= \, ?}{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n), \mathbb{Z}/(m))} \to \mathbb{Z}/(m) \xrightarrow{\cdot n} \mathbb{Z}/(m) \to ?$$

$\varphi : \mathbb{Z}/(n) \to \mathbb{Z}/(m)$
$1 \longmapsto a$
$0 = n \longmapsto n \cdot a = 0$ i.e. $na \in (m)$ i.e. $a \in (m:n)$

$$\Rightarrow a \in \left(\frac{m}{\gcd(m,n)}\right)$$

Ex $n = 12$ $m = 15$ $12 \cdot a \in (15) \Rightarrow a \in (5) = \left(\frac{15}{3}\right)$

$$\Rightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n), \mathbb{Z}/(m)) \approx \mathbb{Z}/\gcd(n,m)$$

continue

$$\mathrm{coker}\left(\mathbb{Z}/(m) \xrightarrow{n} \mathbb{Z}/(m)\right) \simeq \mathbb{Z}/(m) \big/ \gcd(n,m)\mathbb{Z}/(m) \simeq \mathbb{Z}/(\gcd(n,m))$$

$$0 \to \mathbb{Z}/(3) \xrightarrow{\cdot 5} \mathbb{Z}/(15) \xrightarrow{\cdot 12} \mathbb{Z}/(15) \to \mathbb{Z}/(3) \to 0$$