Localization:

Motivation: Construct $\mathbb{Q}$ from $\mathbb{Z}$

(Divide by $\mathbb{Z}-\{0\}$)

General construction in a domain:

$R \supseteq S$
domain, multiplicative subset (closed, 1)

Define equivalence relation on $R \times S$

$(x,s) \sim (y,t)$ if $xt = ys$

(Think of fractions: $\frac{x}{s} = \frac{y}{t}$ if $xt = sy$)

Equivalence relation:

R: $(x,s) \sim (x,s)$ since $xs = xs$

S: $(x,s) \sim (y,t) \iff (y,t) \sim (x,s)$

T: $(x,s) \sim (y,t)$ and $(y,t) \sim (z,u)$, then $xt = sy$, $yu = zt$

and therefore $xut = usy = szt \Rightarrow xu = sz$

since R is a domain.

If R is not domain: Modify relation:

$(x,s) \sim (y,t)$ if $\exists v \in S$ s.t. $xtv = syv$

R: Use $1 \in S$ S: ok T: $\underbrace{xtuv_1v_2}_{xu(tv_1v_2)} = sguv_1v_2 = \underbrace{szt v_1v_2}_{sz(tv_1v_2)}$

where $(x,s) \sim (y,t)$ by v_1 and $(y,t) \sim (z,u)$ by v_2

Notation: $R \times S/_{\sim} =: S^{-1}R$ (Ring: unit $(1,1)$, zero $(0,1)$)

Map $\varphi_S : R \longrightarrow S^{-1}R$ given by $r \mapsto (r,1) = \frac{r}{1}$

Notice: 1) $S \subseteq$ all non-zero divisors, then $\varphi_S : R \to S^{-1}R$ is injective

$\varphi_S(x) = 0 := \frac{0}{1}$ means $\exists u \in S$ s.t. $ux \cdot 1 = u \cdot 1 \cdot 0 = 0$

but u is non-zero-divisor $\Rightarrow x = 0$.

2) S contains zero-divisors: Ex $R = \mathbb{Z}/(6)$, $S = \{1,3\}$

Then $S^{-1}R \simeq \mathbb{Z}/(2)$ since we have

$\frac{0}{1} = \frac{0}{3} = \frac{2}{1} = \frac{2}{3} = \frac{4}{1} = \frac{4}{3}$, $\frac{1}{1} = \frac{1}{3} = \frac{3}{1} = \frac{3}{3} = \frac{5}{1} = \frac{5}{3}$

e.g. $\frac{4}{3} = \frac{2}{1}$ since $(4 \cdot 1 - 2 \cdot 3)3 = 0$.

Ex $S < \mathbb{Z}$, $S = \{1, p, p^2, p^3, \dots\}$

Then $S^{-1}\mathbb{Z} = \mathbb{Z}[\frac{1}{p}] = \{\frac{a}{p^n}\}$ $a \in \mathbb{Z}, n \in \mathbb{Z}$.

Ex $\underset{\text{domain}}{R}$ $S = R \setminus \{0\}$ $\Rightarrow$ $S^{-1}R = \mathrm{Frac}(R)$.

UMP:

$S \subseteq R$ multiplicative subset

$$\begin{tikzcd} S^{-1}R \arrow[r, "\exists! \bar{\varphi}"] & R' \\ R \arrow[u] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

s.t. $\varphi(s)$ is a unit in R' $\forall s \in S$.

Pf. Define $\bar{\varphi}: S^{-1}R \to R'$ by $\bar{\varphi}(x,s) = \varphi(x)\cdot\varphi(s)^{-1}$

Well-defined since $\varphi(s)$ is a unit, and further

$(x,s) \sim (y,t) \overset{\text{def}}{\Longleftrightarrow} \exists u \in S$ s.t. $xtu = syu$

$$\begin{aligned} \Rightarrow \bar{\varphi}(y,t) - \bar{\varphi}(x,s) &= \varphi(y)\cdot\varphi(t)^{-1} - \varphi(x)\cdot\varphi(s)^{-1} \\ &= \varphi(y)\varphi(s)\varphi(s)^{-1}\varphi(t)^{-1} - \varphi(x)\varphi(t)\varphi(t)^{-1}\varphi(s)^{-1} \\ &= \varphi(ys - xt)\cdot\varphi(s)^{-1}\varphi(t)^{-1} \\ &= \varphi((ys-xt)u)\cdot\varphi(u)^{-1}\cdot\varphi(s)^{-1}\varphi(t)^{-1} = 0 \end{aligned}$$

(unique since defined by φ).

Ex 1) $\underset{\text{prime}}{\mathfrak{p}} < R$ $\Rightarrow$ $S = R - \mathfrak{p}$ multiplicative subset $S^{-1}R = R_{\mathfrak{p}}$ (localization in $\mathfrak{p}$)

local ring, max ideal $\varphi_S(\mathfrak{p})\cdot S^{-1}R$

2) $f \in R$, $S = \{1, f, f^2, \dots\}$ $\Rightarrow$ $S^{-1}R = R_f$ (localization in f)

$\cong R[X]/(1 - Xf)$

Pf. We have,

$$\begin{tikzcd} R_f \arrow[r, "\exists! \text{ by UMP}"] & R[X]/(1-fX) \\ R \arrow[u] \arrow[ur] & \end{tikzcd}$$

$f \mapsto f$ and $Xf = 1$.

Define $R[X]/(1-fX) \to R_f$ by $X \mapsto \frac{1}{f}$. Gives iso

$S \subset R \supseteq \mathfrak{a}$
mult.

$\varphi_S : R \to S^{-1}R$

1) $\mathfrak{a}^e = \varphi_S(\mathfrak{a}) \cdot S^{-1}R = S^{-1}\mathfrak{a} = \{\frac{a}{s} \mid a \in \mathfrak{a}, s \in S\}$

2) $\mathfrak{a} \cap S \neq \emptyset \iff \mathfrak{a}^e = (1) \iff (\mathfrak{a}^e)^c = (1)$

Pf. 1) Define $\varphi_S(\mathfrak{a}) \cdot S^{-1}R \longrightarrow S^{-1}\mathfrak{a}$

$$\frac{a}{1} \cdot \frac{x}{s} \longmapsto \frac{ax}{s}$$

and $S^{-1}\mathfrak{a} \longrightarrow \varphi_S(\mathfrak{a}) \cdot S^{-1}R$

$$\frac{a}{s} \longmapsto \frac{a}{1} \cdot \frac{1}{s}$$

2) $s \in \mathfrak{a} \cap S \Rightarrow 1 = \varphi_S(s) \cdot \frac{1}{s} \in \mathfrak{a}^e \Rightarrow \varphi_S^{-1}(\mathfrak{a}^e) = (1)$

$\varphi_S^{-1}(\mathfrak{a}^e) = 1 \Rightarrow \varphi_S(1) \in \mathfrak{a}^e$ i.e. $\frac{1}{1} = \frac{a}{s} \cdot \frac{x}{t}$ i.e. $\exists u \in S$ s.t. $ust = uax$, But $ust \in S$ and $uxa \in \mathfrak{a}$

$\Rightarrow \mathfrak{a} \cap S \neq \emptyset$.

LATER SEE:

$\mathfrak{p}R_{\mathfrak{p}} \subset R_{\mathfrak{p}}$
max. ideal / local ring

In $R_{\mathfrak{p}}$ are all $S = R - \mathfrak{p}$ units i.e. $\mathfrak{p}R_{\mathfrak{p}}$ are the non-units $\Rightarrow$ $R_{\mathfrak{p}}$ local ring.

More general: Ideals in $S^{-1}R$:

Notation: $\mathfrak{a} \subset R$ ideal $\quad \mathfrak{a}^S = \{a \in R \mid \exists s \in S \text{ s.t. } as \in \mathfrak{a}\}$

Ex $(12) \subset \mathbb{Z}$ $\quad S = \{1, 3, 9, 27, \dots\} = [3]$

$\Rightarrow (12)^{[3]} = \{a \in \mathbb{Z} \mid \exists\, 3^n \text{ s.t. } 12 \mid a 3^n\} = (4)$

$(12)^{[5]} = (12)$

$(12)^{[2]} = (3)$

Lemma $\mathfrak{p} \subseteq R \supseteq S$ (prime, m.c), $\mathfrak{p} \cap S = \emptyset$

a) $\mathfrak{p} = \mathfrak{p}^S = \{x \in R \mid \exists s \in S \text{ with } xs \in \mathfrak{p}\}$

b) $\mathfrak{p} S^{-1}R$ prime

Pf. a) $\mathfrak{p} \subset \mathfrak{p}^S$ since $1 \in S$

$x \in \mathfrak{p}^S \Rightarrow xs \in \mathfrak{p}$ i.e. $x \in \mathfrak{p}$ since $s \in S \subset R - \mathfrak{p}$.

b) Suppose $\frac{a}{s} \cdot \frac{b}{t} \in \mathfrak{p} S^{-1}R$. Then $ab \in \varphi_S^{-1}(\mathfrak{p} S^{-1}R)$

But $\varphi_S^{-1}(\mathfrak{p} S^{-1}R) = \mathfrak{p}$. Why? We have $\mathfrak{p} \subset \varphi_S^{-1}(\mathfrak{p} S^{-1}R)$

Pick $x \in \varphi_S^{-1}(\mathfrak{p} S^{-1}R) \Rightarrow \frac{x}{1} \in \mathfrak{p} S^{-1}R$

$\Rightarrow \frac{x}{1} \sim \frac{a}{s}$, $a \in \mathfrak{p}$, $s \in S \Rightarrow uxs = ua \in \mathfrak{p}$

Since $\mathfrak{p}$ is prime and $us \in S$, $\mathfrak{p} \cap S = \emptyset$

it follows that $x \in \mathfrak{p}$.

Thus $a \in \varphi_S^{-1}(\mathfrak{p} S^{-1}R)$ or $b \in \varphi_S^{-1}(\mathfrak{p} S^{-1}R)$

and $\varphi(a) \in \mathfrak{p} S^{-1}R$ or $\varphi(b) \in \mathfrak{p} S^{-1}R$ i.e. $\frac{a}{s} \in \mathfrak{p} S^{-1}R$

or $\frac{b}{t} \in \mathfrak{p} S^{-1}R$

Prop. $R \supseteq S$ (m.c.)

$\mathfrak{p} \longmapsto \mathfrak{p} S^{-1}R$ — inclusion-presery.

$\varphi_S^{-1}(\mathfrak{q}) \longleftarrow \mathfrak{q}$ — bijection

s.t. $\mathfrak{p} \cap S = \emptyset$.

Prop. $\mathfrak{p} R_{\mathfrak{p}} \subseteq R_{\mathfrak{p}}$ (max, local)

Pf $\mathfrak{q} \subsetneq R_{\mathfrak{p}}$ (ideal). Then $\varphi_{\mathfrak{p}}^{-1}(\mathfrak{q}) \subset \mathfrak{p}$ and

$\mathfrak{q} \subset \varphi_{\mathfrak{p}}(\mathfrak{p})$ ← only max ideal

If $\frac{x}{s} \in R_{\mathfrak{p}}$ is a unit. Then $\exists \frac{y}{t}$ s.t. $\frac{xy}{st} = 1$

$\Rightarrow \exists u \notin \mathfrak{p}$ s.t. $xyu = stu \notin \mathfrak{p} \Rightarrow x \notin \mathfrak{p}$

Prop. $R \supseteq \underset{\text{m.c.}}{S}$ $\qquad S^{-1}R[X] = S^{-1}(R[X])$

Pf

$$\begin{tikzcd}
 & S^{-1}(R[X]) \arrow[dr, "\exists!"] & \\
S^{-1}R[X] \arrow[rr, "\exists!"'] & & R' \\
R \arrow[u] \arrow[uur] \arrow[urr, "\psi"'] & &
\end{tikzcd}$$

$\psi(S)$ units in R'

Assum $X \mapsto x$.

Why

1)
$$\begin{tikzcd}
 & S^{-1}R[X] \arrow[d, "\exists!"] \\
S^{-1}R \arrow[ur] \arrow[r, "\exists!"] & R' \\
R \arrow[u, "\varphi_S"] \arrow[ur] &
\end{tikzcd}$$

2)
$$\begin{tikzcd}
 & S^{-1}R[X] \arrow[d, "\exists!"] \\
R[X] \arrow[ur] \arrow[r, "\exists!"] & R' \\
R \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Both satisfy UMP, by universality they are iso