Def

$S \subset R$ mc (ring) $\quad M$ R-module $\qquad S^{-1}M := M \times S/_{\sim}$

where $\sim$: $(m,s) \sim (n,t)$ if $\exists u \in S$ s.t. $(mt - sn)u = 0$

(equivalence relation)

Notice: • $\varphi_S : M \longrightarrow S^{-1}M$, $m \longmapsto \frac{m}{1}$ $\quad$ R-linear

• $\mu_s : S^{-1}M \xrightarrow{\cdot \frac{s}{1}} S^{-1}M$ $\quad$ bijective $\forall s \in S$

Pf injective: $\frac{m}{t} \cdot \frac{s}{1} = \frac{ms}{t} = \frac{0}{1}$, i.e. $\exists u \in S$ s.t.

$ums = us \cdot m = 0$

But the $\exists us \in S$ s.t. $\frac{m}{t} = \frac{0}{1}$ $\quad (m \cdot 1 - t \cdot 0)us = 0$

surjective: $\frac{1}{s} \cdot \frac{m}{t} \longmapsto \frac{m}{t}$

• $M_f := S^{-1}M$ with $S = \{1, f, f^2, \dots\}$

• $M_{\mathfrak{p}} := S^{-1}M$ with $S = R \setminus \mathfrak{p}$.

Functoriality:

•
$$\begin{tikzcd} M \arrow[r, "\varphi_S"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circ"] & S^{-1}M \arrow[d, "S^{-1}\alpha"] \\ N \arrow[r, "\varphi_S"'] & S^{-1}N \end{tikzcd}$$
R-linear

• $\mathrm{Hom}_R(M, f(N)) = \mathrm{Hom}_{S^{-1}R}(S^{-1}M, N)$ $\qquad$ M R-mod $\quad N$ $S^{-1}R$-mod

$\rightarrow$ by UMP

$\leftarrow$ restriction of scalars

f: forgetfull functor (restriction of scalar)

Prop. $S \subset R$ mc, M R-module $\Rightarrow$ $S^{-1}R \otimes_R M \xrightarrow{\sim} S^{-1}M$

$$\frac{a}{s} \otimes m \longmapsto \frac{am}{s} \qquad \forall a \in R, s \in S, m \in M$$

Pf.

- R-linear
- Injective: $\frac{am}{s} = 0 \Leftrightarrow \exists u \in S$ s.t. $uam = 0$

  i.e. $\frac{a}{s} \otimes m = \frac{u}{u}\frac{a}{s} \otimes m = \frac{1}{us} \otimes uam = 0$

  We have $\sum \frac{a_i m_i}{s_i} = \sum \frac{a_i m_i \cdot \frac{\prod s_i}{s_i}}{\prod s_i} = \frac{1}{s} \sum a_i r_i m_i$

  where $s = \prod s_i$, $r_i = \frac{s}{s_i}$. Then

  $$\sum_i \frac{a_i}{s_i} \otimes m_i = \frac{1}{s} \otimes \sum_i a_i r_i m_i = \frac{1}{s} \otimes m.$$

- Surjective: $\frac{1}{s} \otimes m \mapsto \frac{m}{s}$.

Prop. S^{-1} is exact, i.e. $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ exact

$$\Rightarrow S^{-1}M' \xrightarrow{S^{-1}\alpha} S^{-1}M \xrightarrow{S^{-1}\beta} S^{-1}M'' \text{ exact.}$$

Pf. Let $S^{-1}\beta\left(\frac{m}{s}\right) = \frac{\beta(m)}{s} = 0$. Then $\exists u \in S$ s.t. $\beta(m) \cdot u = 0$

Thus $\beta(mu) = 0$ and $\exists\, m' \in M$ s.t. $\alpha(m') = mu$. But

then $S^{-1}\alpha\left(\frac{m'}{us}\right) = \frac{\alpha(m')}{us} = \frac{mu}{su} = \frac{m}{s}$.

$\Longrightarrow$ $S^{-1}R$ is flat over R

Pf $0 \to M' \to M \to M'' \to 0$ exact.

$\Rightarrow 0 \to S^{-1}M' \to S^{-1}M \to S^{-1}M'' \to 0$ exact

$\Rightarrow 0 \to S^{-1}R \otimes_R M' \to S^{-1}R \otimes_R M \to S^{-1}R \otimes_R M'' \to 0$ exact.

Local properties:

R ring, M, N module

1) $M=0 \iff \underset{\overset{\shortparallel}{M_{\mathfrak{p}}}}{M \otimes_R R_{\mathfrak{p}}}=0 \ \forall$ prim $\mathfrak{p} \iff M_{\underline{m}}=0 \ \forall$ max $\underline{m}$

Pf. Suppose $M \neq 0$, and let $0 \neq x \in M$. Let $\mathfrak{a} = Ann(x)$, ideal

$\Rightarrow \mathfrak{a} \subseteq \underline{m}$ for some max ideal, and $\frac{x}{1} \in M_{\underline{m}} = 0$ by assumpta

i.e. $\exists u \in R - \underline{m}$ s.t. $ux=0$. But then $u \in Ann(x) \subseteq \underline{m}$

and $u \notin m$. Contradiction.

2) $M \overset{\varphi}{\hookrightarrow} N \iff M_{\mathfrak{p}} \hookrightarrow N_{\mathfrak{p}} \iff M_{\underline{m}} \hookrightarrow N_{\underline{m}}$

Pf Suppose $\ker(\varphi) \neq 0$, then $0 \to \ker\varphi \to M \to N$ exact

$\Rightarrow 0 \to \ker\varphi \otimes S^{-1}R \longrightarrow M_{\underline{m}} \to N_{\mathfrak{p}}$ exact

$\Rightarrow \quad \ker(\varphi_{\underline{m}}) \simeq \ker(\varphi) \otimes_R S^{-1}R = 0$ by assumption

$\Rightarrow \quad \ker(\varphi) = 0$ by. 1).

3) M flat$/R \iff M_{\mathfrak{p}}$ flat$/R_{\mathfrak{p}} \iff M_{m}$ flat$/M_m$

Pf. $N \hookrightarrow P \Rightarrow N_{\underline{m}} \hookrightarrow P_{\underline{m}}$

$\Rightarrow N_{\underline{m}} \otimes_{A_m} M_m \hookrightarrow P_{\underline{m}} \otimes_{A_{\underline{m}}} M_m$

$\Rightarrow (N \otimes_A M)_m \hookrightarrow (P \otimes_A M)_{\underline{m}}$

$\Rightarrow N \otimes_A M \hookrightarrow P \otimes_A M$

Prop.

$S \subset R$ mc M, N R-modules $\sigma : S^{-1}\mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$ homomorphism

M fin. gen $\Rightarrow$ σ injective.

M fin. presented $\Rightarrow$ σ iso.

Pf. $\varphi \in \mathrm{Hom}(M,N)$, $s \in S$; $\frac{\varphi}{s} \in S^{-1}\mathrm{Hom}_R(M,N)$

Define
$$\sigma\left(\frac{\varphi}{s}\right)\left(\frac{m}{u}\right) = \frac{\varphi(m)}{su}.$$

$S^{-1}R$-linear since φ is R-linear

For $M = R$ we have $\sigma : S^{-1}\mathrm{Hom}_R(R,N) = S^{-1}N \to \mathrm{Hom}_{S^{-1}R}(S^{-1}R, S^{-1}N)$
and σ is a bijection $\cong S^{-1}N$
If M is fin gen we have $R^{\oplus \Lambda} \to R^n \to M \to 0$ exact

$\Rightarrow$
$$\begin{tikzcd}
 & S^{-1}N^n & & \\
S^{-1}\mathrm{Hom}(R^{\oplus\Lambda}, N) \arrow[d, "\theta"] & S^{-1}\mathrm{Hom}_R(R^n, N) \arrow[l] \arrow[d, "\cong"] & S^{-1}\mathrm{Hom}(M,N) \arrow[l] \arrow[d, "\sigma"] & 0 \arrow[l] \\
\mathrm{Hom}_{S^{-1}R}(S^{-1}R^{\oplus\Lambda}, S^{-1}N) & \mathrm{Hom}_{S^{-1}R}(S^{-1}R^n, N) \arrow[l] & \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N) \arrow[l] & 0 \arrow[l] \\
 & S^{-1}N^n \arrow[u, phantom, sloped, "\cong"] & &
\end{tikzcd}$$

Middle map is iso $\Rightarrow$ σ injective.

θ is iso $\Rightarrow$ σ is iso

Ex. $S = \mathbb{Z} \setminus \{0\} \subseteq \mathbb{Z}$ $\quad M = \mathbb{Q}/\mathbb{Z}$.
mc

$S^{-1}\mathbb{Z} = \mathbb{Q}$!!

$$\sigma : S^{-1} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \longrightarrow \operatorname{Hom}_{\mathbb{Q}}(S^{-1}M, S^{-1}M)$$

$$\uparrow \text{injective } (*)$$

$$\mathbb{Z}.$$

Pf of $(*)$: We have $\frac{n}{1} = 2n \cdot \frac{1}{2} \neq 0$ since $\frac{1}{2} \notin \mathbb{Z}$. On the other hand $S^{-1}M = 0$ since $s \cdot \frac{r}{s} = 0 \;\forall\; \frac{r}{s} \in \mathbb{Q}/\mathbb{Z}$.

Thus σ is non injective.