$$\mathrm{Spec}(R) = \{ \mathfrak{p} \subset R \text{, prime} \}.$$

prim spectrum

$$V(\mathfrak{a}) = \{ \mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{a} \subset \mathfrak{p} \}$$

Notice 1) $\underset{\sqrt{\mathfrak{a}}}{\overset{\shortparallel}{r(\mathfrak{a})}} = \bigcap\limits_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$ $\Rightarrow$ $V(\mathfrak{a}) = V(\mathfrak{b})$ iff $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$

Pf $\Rightarrow$ obvious

$\Leftarrow$ $\mathfrak{a} \subset \mathfrak{p} \Rightarrow \sqrt{\mathfrak{a}} \subset \mathfrak{p}$.

2) $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a} \cdot \mathfrak{b})$

3) $\bigcap\limits_{\lambda} V(\mathfrak{a}_\lambda) = V(\Sigma \mathfrak{a}_\lambda)$

4) $V(R) = \emptyset$ $\quad V((0)) = \mathrm{Spec}(R)$

$\Rightarrow$ $\{ V(\mathfrak{a}) \}$ gives the closed sets of a topology on $\mathrm{Spec}(R)$.

Zariski topology

Basis of Open sets : $D(f) := \mathrm{Spec}(R) \setminus V((f))$

$\underset{\text{prime}}{\mathfrak{p}} \not\supset \mathfrak{a} \Rightarrow \exists f \in \mathfrak{a} - \mathfrak{p}$ and $\mathfrak{p} \in D(f)$.

$$D(f) \cap D(g) = D(fg)$$

$\varphi : R \to R'$ induces $\varphi^a : \mathrm{Spec}(R') \to \mathrm{Spec}(R)$

$\mathfrak{p} \longmapsto \varphi^{-1}(\mathfrak{p})$

Ex. $\varphi : R \to R/\mathfrak{a}$ induces $\varphi^a : \mathrm{Spec}(R/\mathfrak{a}) \to \mathrm{Spec}(R)$

$\mathrm{im}\, \varphi^a = V(\mathfrak{a})$ closed

$\varphi_f : R \to R_f$ induces $\varphi_f^a : \mathrm{Spec}(R_f) \to \mathrm{Spec}\, R$

$\mathrm{Im}\, \varphi_f^a = D(f)$ open

Prop $X = \mathrm{Spec}(R)$ is quasi-compact.

Pf. Let $X = \bigcup_\lambda U_\lambda$, $U_\lambda = \mathrm{Spec}(R) \setminus V(\mathfrak{a}_\lambda)$

$\Rightarrow V(\sum \mathfrak{a}_\lambda) = \bigcap_\lambda V(\mathfrak{a}_\lambda) = \emptyset$ since $\bigcup(X - V(\mathfrak{a}_\lambda)) = X \setminus \bigcap V(\mathfrak{a}_\lambda) = X$.

$\Rightarrow \sum \mathfrak{a}_\lambda = (1)$ i.e. $\exists$ finite set $\sum_i f_i = 1$

$\Rightarrow \sum_i \mathfrak{a}_i = (1) \Rightarrow V(\sum_i \mathfrak{a}_i) = \emptyset \Rightarrow \bigcap_i V(\mathfrak{a}_i) = \emptyset$

$\Rightarrow \bigcup_i X \setminus V(\mathfrak{a}_i) = X$.

Definition

R ring, M R-module

$\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_\mathfrak{p} \neq 0\}$

Support of M

Prop: 1) $0 \to L \to M \to N \to 0$ R-exact $\Rightarrow$ $\mathrm{Supp}(L) \cup \mathrm{Supp}(N) = \mathrm{Supp}(M)$

Pf. By local property.

2) $\sum M_\lambda = M \Rightarrow \bigcup \mathrm{Supp}(M_\lambda) = \mathrm{Supp}(M)$

Pf. $M_\lambda \subset M$, by 1) $\mathrm{Supp}(M_\lambda) \subset \mathrm{Supp}(M)$

If $\mathfrak{p} \notin \bigcup \mathrm{Supp}(M_\lambda)$, then $(M_\lambda)_\mathfrak{p} = 0$ $\forall \lambda$.

But $\sum M_\lambda = M$ implies $\bigoplus M_\lambda \twoheadrightarrow M$ (surjective)

$\Rightarrow \bigoplus (M_\lambda)_\mathfrak{p} \twoheadrightarrow M_\mathfrak{p} \Rightarrow M_\mathfrak{p} = 0$

3) $\mathrm{Supp}(M) \subset V(\mathrm{Ann}(M))$ equality if M is fin. gen

Pf. $M_\mathfrak{p} = 0$ iff $\mathrm{Ann}(M) \cap (R - \mathfrak{p}) = \emptyset$.

[$x \in \mathrm{Ann}(M)$, $x \in R - \mathfrak{p} \Rightarrow \exists x \in S$ s.t. $sm = 0$, i.e. $\frac{m}{1} = 0$]

M fin. gen $= \langle m_1, \dots, m_n \rangle \Rightarrow \exists f_i \in R \setminus \mathfrak{p}$ s.t. $f_i m_i = 0$

$\Rightarrow \prod f_i \cdot M = 0$ and $\prod f_i \in R - \mathfrak{p}$

Prop. $\mathrm{Supp}(M \otimes_R N) \subset \mathrm{Supp}(M) \cup \mathrm{Supp}(N)$

fin.gen

Lemma 1 R local ring. $M \otimes_R N \neq 0$ if $M \neq 0$ and $N \neq 0$

Pf. We have by Nakayamas Lemma

$$M \otimes k = M/\mathfrak{m}M = 0 \Leftrightarrow M = 0$$

But $M, N \neq 0 \Rightarrow M \otimes k, N \otimes k \neq 0$

$$\Rightarrow 0 \neq (M \otimes_R k) \otimes_R (N \otimes k) \simeq (M \otimes N) \otimes (k \otimes_R k) \simeq (M \otimes N) \otimes k$$

$$\Rightarrow M \otimes N \neq 0$$

Lemma 2

$$S^{-1}(M \otimes_R N) = S^{-1}M \otimes_R N = S^{-1}M \otimes_{S^{-1}R} S^{-1}N = S^{-1}M \otimes_R S^{-1}N.$$

Pf. $S^{-1}(M \otimes_R N) = S^{-1}R \otimes_R (M \otimes_R N) = (S^{-1}R \otimes_R M) \otimes_R N = S^{-1}M \otimes_R N$

$S^{-1}M \otimes_R N = S^{-1}M \otimes_{S^{-1}R} S^{-1}R \otimes_R N = S^{-1}M \otimes_{S^{-1}R} S^{-1}N$

$\varphi : S^{-1}M \otimes_R S^{-1}N \longrightarrow S^{-1}M \otimes_{S^{-1}R} S^{-1}N.$

$\Rightarrow \ker \varphi = \langle \frac{xm}{s} \otimes \frac{n}{1} - \frac{m}{1} \otimes \frac{xn}{s} \rangle$

But $\mu_s : S^{-1}M \otimes_R S^{-1}N \circlearrowleft$ bijection.

$\Rightarrow s\left(\frac{xm}{s} \otimes \frac{n}{1} - \frac{m}{1} \otimes \frac{xn}{s}\right) = \frac{xm}{1} \otimes \frac{n}{1} - \frac{m}{1} \otimes \frac{xn}{1} = 0.$

Pf.

$$(M \otimes_R N)_{\mathfrak{p}} = M_{\mathfrak{p}} \otimes_R N_{\mathfrak{p}} \neq 0 \overset{2}{\Rightarrow} M_{\mathfrak{p}} \neq 0 \text{ or } N_{\mathfrak{p}} \neq 0$$

$$N_{\mathfrak{p}} \neq 0 \text{ or } M_{\mathfrak{p}} \neq 0 \overset{1}{\Rightarrow} M_{\mathfrak{p}} \otimes_R N_{\mathfrak{p}} \neq 0 \Rightarrow (M \otimes_R N)_{\mathfrak{p}} \neq 0$$

Notice

$$\mathrm{Supp}(M) = \emptyset \Leftrightarrow M_{\mathfrak{p}} = 0 \quad \forall \mathfrak{p} \in \mathrm{Spec}(R)$$

Ex. $R = k[x,y]$ Ideal $\mathfrak{p} = (x^2+y^2-1)$ $M = k[x,y]/(x^2+y^2-1)$

$k = \bar{k}$ R-module

Max ideal $\underline{m} = \langle x-a, y-b \rangle$.

Then $\mathfrak{p} \subseteq \underline{m}$ iff. $a^2+b^2-1=0$

$$x^2+y^2-1 = f(x,y)(x-a) + g(x,y)(y-b) + \text{rest}$$
$$= (x+a)(x-a) + (y+b)(y-b) + (a^2+b^2-1)$$
$$\Rightarrow x^2+y^2-1 \in (x-a, y-b) \text{ iff } a^2+b^2-1=0$$

$M_{\underline{m}} \neq 0 \Rightarrow \mathrm{Ann}(M) \cap R \setminus \underline{m} = \emptyset$

But $\mathrm{Ann}\left(R/(x^2+y^2-1)\right) = (x^2+y^2-1) = \mathfrak{p}$

i.e. $\mathfrak{p} \cap (R \setminus \underline{m}) = \emptyset$ i.e. $\mathfrak{p} \subseteq \underline{m}$

$\Leftrightarrow a^2+b^2-1=0$

$$\mathrm{Supp}(M) = \{ (x-a, y-b) \mid a^2+b^2-1=0 \}.$$