11.2 $S \subset R$ $S^{-1}R = 0 \iff S \cap \sqrt{(0)} \neq \emptyset$

m.c. ring

Pf. $S^{-1}R = 0 \iff 0 = \frac{1}{1} \in R \iff \exists u \in S$ s.th. $u = 0$

$\iff 0 \in S \iff \exists s \in S$ s.t. $s^n = 0$ for some n

($\Leftarrow$ by m.c. condition)

11.5 $\mathbb{Z} \subset R \subset \mathbb{Q}$

$$\mathbb{Z}[\tfrac{2}{3}] = \left\{ \frac{q}{3^n},\ n \geq 0\ q \in \mathbb{Z} \right\} = S^{-1}\mathbb{Z} \quad S = \{3^i \mid i \geq 0\}.$$

Consider the set $\left\{ y \mid \frac{x}{y} \in R,\ \gcd(x,y) = 1 \right\} = S_R$

- mult. closed: Consider $\frac{x_1}{y_1}, \frac{x_2}{y_2}$, $\gcd(x_1,y_2) = \gcd(x_2,y_1) = 1$

  then $\frac{x_1}{y_1} \cdot \frac{x_2}{y_2} = \frac{x_1x_2}{y_1y_2}$ and $\gcd(x_1x_2, y_1y_2) = 1$

- $1 \in S_R$
- $S_R^{-1}\mathbb{Z} = R$ $\subseteq$: obvious.

  $\supseteq$: $\frac{x}{y} \in R$, $\gcd(x,y) = 1$, then $y \in S_R$

  and $x \cdot \frac{1}{y} = \frac{x}{y} \in S_R^{-1}\mathbb{Z}$

- Let $p \mid y$, $y \in S_R$. Then, with $q = \frac{y}{p}$, we have

  $\frac{x}{y} \cdot q = \frac{x}{p} \in R$.

  i.e. $S_R \longleftrightarrow$ {subset of primes}.

11.8 R', R'' $S = \{(1,1), (1,0)\} \Rightarrow S^{-1}R = R'$

$R = R' \times R''$

Pf. We have $R' \to R' \times R''$, $a \mapsto (a,0)$

Injective since $(a,0) = 0 \in S^{-1}R$ iff $(1,0)(a,0) = 0$

i.e. $a = 0$.

Surjective: $(a,x) \in R$, then $(a,x) = (a,0)$

since $(1,0)(a,x) = (1,0)(a,0) = (a,0)$.

11.16 $\underset{mc}{S} \subset R \qquad nil(R) \cdot S^{-1}R = nil(S^{-1}R)$

Pf. $\subseteq$ $x \in R$, $x^n = 0$, $\frac{r}{s} \in S^{-1}R$, then

$$\left(\frac{xr}{s}\right)^n = x^n \cdot \left(\frac{r}{s}\right)^n = 0.$$

$\supseteq$ $\left(\frac{x}{s}\right)^n = 0 \Rightarrow \exists u \in S$ s.t. $u \cdot x^n = 0$.

$\Rightarrow \frac{x}{s} = ux \cdot \frac{1}{us} \in nil(R) \cdot S^{-1}R$

since $(ux)^n = u^n \cdot x^n = u^{n-1} u x^n = 0$.

11.28 $\underset{mc}{S, T} \subset R$

1) $S \subset T$, $T' = \varphi_S(T) \Rightarrow T^{-1}R = T'^{-1}(S^{-1}R) = T^{-1}(S^{-1}R)$

2) $U = S \cdot T \Rightarrow T^{-1}(S^{-1}R) = S^{-1}(T^{-1}R) = U^{-1}R$.

3) $S' = \bigcup_{t \in R} R(S : t) \Rightarrow S'^{-1}R = S^{-1}R$.

Pf 1) $\frac{r}{t} = \frac{rs}{ts} = \frac{\frac{1}{1}}{\frac{t}{s}} \cdot \frac{r}{s} = \frac{1}{t} \cdot \frac{r}{1}$

2) $\frac{1}{t} \cdot \frac{r}{s} = \frac{1}{s}\frac{r}{t} = \frac{r}{st}$

3) $\frac{r}{s'} = \frac{qr}{qs'} = \frac{qr}{s}$ where $s' \cdot q \in S$. $s' \in (S : q)$

but $S \subset S'$ since $s \cdot 1 \in S$.

12.4 $\underset{mc}{S} \subset R \qquad M = S^{-1}M \Leftrightarrow \underset{S^{-1}R\text{-module}}{M}$

$\underset{R\text{-module}}{M}$ $\qquad \Rightarrow$ Obvious

$\Leftarrow$
$$\begin{tikzcd}
S^{-1}M \arrow[r, "\exists!"] & M' \\
M \arrow[u, "\varphi_S"] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$
M' $S^{-1}R$-module with φ R-linear

Then

$$\begin{tikzcd} S^{-1}M \arrow[r, "\exists!"] & M \\ M \arrow[u, "\varphi_S"] \arrow[ur, "1"'] & \end{tikzcd}$$

$S^{-1}R$-module
with 1 R-linear

$\Rightarrow \quad S^{-1}M \simeq M$.

12.5 $S \subset T \subset R$

M
R-module.

$$T^{-1}M = T^{-1}(S^{-1}M) = T_1^{-1}(S^{-1}M)$$

$T_1 = \varphi_S(T) \subset S^{-1}R$

Pf

$$\begin{tikzcd} & T^{-1}M \\ M \arrow[ur, "\psi"] & \end{tikzcd}$$

ψ_1 : R-linear
$T^{-1}M$: $T_1^{-1}R$-module

$\because$ $T^{-1}M$ has a $S^{-1}R$-module structure since $S \subset T$, hence ψ_1 factors through $S^{-1}M$

$$\begin{tikzcd} T^{-1}R\text{-mod} \ T^{-1}(S^{-1}M) \arrow[r, shift left, "\exists!"] & T^{-1}M \arrow[l, shift left, "\exists!"] \\ S^{-1}M \arrow[u] \arrow[ur, "\psi_2"] & \\ M \arrow[u] \arrow[uur, "\psi_1"'] & \end{tikzcd}$$

ψ_2 : $S^{-1}R$-linea
$T^{-1}M$: $T^{-1}(S^{-1}R)$-module

12.7 $S \subset R$
mc

M
R-module

$S^{-1}M = 0 \Leftarrow \operatorname{Ann}(M) \cap S \neq \emptyset$
$\Rightarrow$ if M is fin gen.

Pf $S^{-1}M = 0 \Rightarrow \forall m \in M \ \exists u \in S$ s.t. $um = 0$

$M = \langle m_1, .., m_r \rangle$, $u_i m_i = 0$
fin gen

$\prod u_i \cdot \sum r_i m_i = \sum r_i \prod_{j \neq i} u_j \, u_i m_i = 0$

$\Rightarrow \prod u_i \in \operatorname{Ann}(M)$ and $\prod u_i \in S$.

Let $x \in \operatorname{Ann}(M) \cap S$, and $\frac{m}{s} \in S^{-1}M$.

Then $\frac{x}{1} \cdot \frac{m}{s} = 0$ i.e. $x \cdot \frac{m}{s} = 0$ or $\frac{x}{x} \frac{m}{s} = \frac{m}{s} = 0$

since $x \in S$.

12.13

$S \subset R$ mc

M, N R-modules

$$S^{-1}(M \otimes_R N) \overset{1)}{=} S^{-1}M \otimes_R N = S^{-1}M \otimes_{S^{-1}R} S^{-1}N = S^{-1}M \otimes_R S^{-1}N$$

Pf 1) $S^{-1}(M \otimes_R N) \approx S^{-1}R \otimes_R (M \otimes_R N) \approx (S^{-1}R \otimes_R M) \otimes_R N$

$$\approx S^{-1}M \otimes_R N$$

$$S^{-1}M \otimes_R N \approx S^{-1}M \otimes_{S^{-1}R} S^{-1}R \otimes_R N = S^{-1}M \otimes_{S^{-1}R} S^{-1}N$$

$$\mathrm{Ker} \hookrightarrow S^{-1}M \otimes_R S^{-1}N \longrightarrow S^{-1}M \otimes_{S^{-1}R} S^{-1}N$$

gen by

$$\frac{xm}{s} \otimes \frac{n}{1} - \frac{m}{1} \otimes \frac{xn}{s} = \frac{s}{s}\left(\frac{xm}{s} \otimes \frac{n}{1} - \frac{m}{1} \otimes \frac{xn}{s}\right)$$

$$= \frac{1}{s}(xm \otimes n - m \otimes xn) = 0$$

12.24

$S = \mathbb{Z} \setminus \{0\} \subset \mathbb{Z}$

$M = \bigoplus_{n \geq 2} \mathbb{Z}/(n)$

$$\varphi : S^{-1}\mathrm{Hom}_{\mathbb{Z}}(M, M) \longrightarrow \mathrm{Hom}_{\mathbb{Q}}(S^{-1}M, S^{-1}M)$$

$$S^{-1}M = \bigoplus_{n \geq 2} S^{-1}\mathbb{Z}/(n)$$

$$S^{-1}\mathbb{Z}/(n) = 0 \Leftarrow \exists u \in S \quad u \cdot 1 = 0.$$

We have $1 \in \mathrm{Hom}_{\mathbb{Z}}(M, M)$ and $\varphi_S(1) = 0$. In fact if $\varphi_S(1) = 0$ then $\exists u \in S$ s.t. $u \cdot 1 = 0$. But $\forall u \in S$, we can find $n \geq 2$ s.t. $u \neq 0$ in $\mathbb{Z}/(n)$.