

$$11.2 \quad S \subset R \quad S^{-1}R = 0 \iff S \cap \sqrt{(0)} \neq \emptyset$$

m.c. mng

PP. $S^{-1}R = 0 \iff 0 = \frac{1}{s} \in R \iff \exists s \in S \text{ s.t. } s \cdot 0 = 0$

$\iff 0 \in S \iff \exists s \in S \text{ s.t. } s^n = 0 \text{ for some } n$

(\Leftarrow by m.c. condition)

$$11.5 \quad \mathbb{Z} \subset R \subset \mathbb{Q}$$

$$\mathbb{Z} \left[\frac{\mathbb{Q}}{\mathbb{Z}} \right] = \left\{ \frac{a}{3^n}, n \geq 0, a \in \mathbb{Z} \right\} = S^{-1}\mathbb{Z} \quad S = \{3^i \mid i \geq 0\}$$

Consider the set $\left\{ \frac{x}{y} \mid \frac{x}{y} \in R, \gcd(x,y) = 1 \right\} = S_R$

- mult. closed: Consider $\frac{x_1}{y_1}, \frac{x_2}{y_2}, \gcd(x_1, y_1) = \gcd(x_2, y_2) = 1$

then $\frac{x_1}{y_1} \cdot \frac{x_2}{y_2} = \frac{x_1 x_2}{y_1 y_2}$ and $\gcd(x_1 x_2, y_1 y_2) = 1$

- $1 \in S_R$

- $S_R^{-1}\mathbb{Z} = R \quad \Leftarrow$: obvious.

\Rightarrow : $\frac{x}{y} \in R, \gcd(x,y) = 1$, then $y \in S_R$

and $x \cdot \frac{1}{y} = \frac{x}{y} \in S_R^{-1}\mathbb{Z}$

- Let $p \mid y, y \in S_R$. Then, with $q = \frac{y}{p}$, we have

$\frac{x}{y} \cdot q = \frac{x}{p} \in R$

i.e. $S_R \iff \{\text{subset of primes}\}$

$$11.8 \quad R^1, R^2 \quad S = \{(1,1), (1,0)\} \quad \Rightarrow \quad S^{-1}R = R^1$$

$R = R^1 \times R^2$

PP We have $R^1 \rightarrow R^1 \times R^2, a \mapsto (a, 0)$

Injective since $(a, 0) = 0 \in S^{-1}R$ iff $(1, 0)(a, 0) = 0$

i.e. $a = 0$.

Surjective: $(a, x) \in R$, then $(a, x) = (a, 0)$

since $(1, 0)(a, x) = (1, 0)(a, 0) = (a, 0)$.

11.16

$$S \subseteq R \quad \text{nil}^n(R) \cdot S^{-1}R = \text{nil}(S^{-1}R)$$

pp. $\subseteq X \in R, x^n = 0, \frac{x}{s} \in S^{-1}R$, then

$$\left(\frac{x}{s}\right)^n = x^n \left(\frac{1}{s}\right)^n = 0.$$

$$\Rightarrow \left(\frac{x}{s}\right)^n = 0 \Rightarrow \exists u \in S \text{ s.t. } u \cdot x^n = 0.$$

$$\Rightarrow \frac{x}{s} = u x \cdot \frac{1}{us} \in \text{nil}(R) \cdot S^{-1}R$$

$$\text{since } (ux)^n = u^n \cdot x^n = u^{n-1} u x^n = 0.$$

11.28

S, T \subseteq R

$$1) \text{ SCT, } T^{-1} = \varphi_S(T) \Rightarrow T^{-1}R = T^{-1}(S^{-1}R) = (T^{-1}S^{-1}R)$$

$$2) U = S \cdot T \Rightarrow T^{-1}(S^{-1}R) = S^{-1}(T^{-1}R) = U^{-1}R$$

$$3) S^{-1} = \bigcup_{t \in R} (S: t) \Rightarrow S^{-1}R = S^{-1}R$$

$$\text{pp } 1) \frac{1}{t} = \frac{1s}{ts} = \frac{1}{t} \cdot \frac{1}{s} = \frac{1}{t} \cdot \frac{1}{s}$$

$$2) \frac{1}{t} \cdot \frac{1}{s} = \frac{1}{s} \cdot \frac{1}{t} = \frac{1}{st}$$

$$3) \frac{1}{s'} = \frac{\varphi(1)}{\varphi(s)} = \frac{\varphi(1)}{s} \quad \text{where } s', \varphi \in S, s' \in (S: \varphi)$$

but $S \subseteq S'$ since $s \cdot t \in S$.

12.4

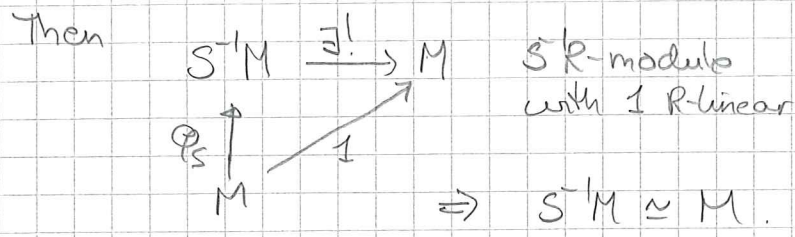
S \subseteq R

mc

$$M = S^{-1}M \Leftrightarrow M \text{ } S^{-1}R\text{-module}$$

M
R-module \Rightarrow Obvious

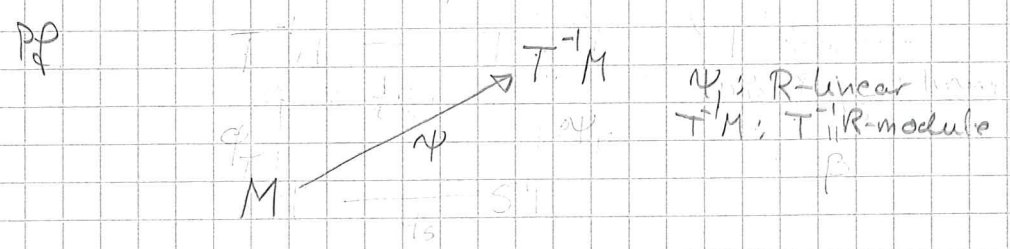
$$\leftarrow \begin{array}{ccc} S^{-1}M & \xrightarrow{\exists!} & M \\ \uparrow \varphi_S & & \uparrow \varphi \\ M & & M \end{array} \quad \begin{array}{l} S^{-1}R\text{-module} \\ \text{with } \varphi \text{ R-linear} \end{array}$$



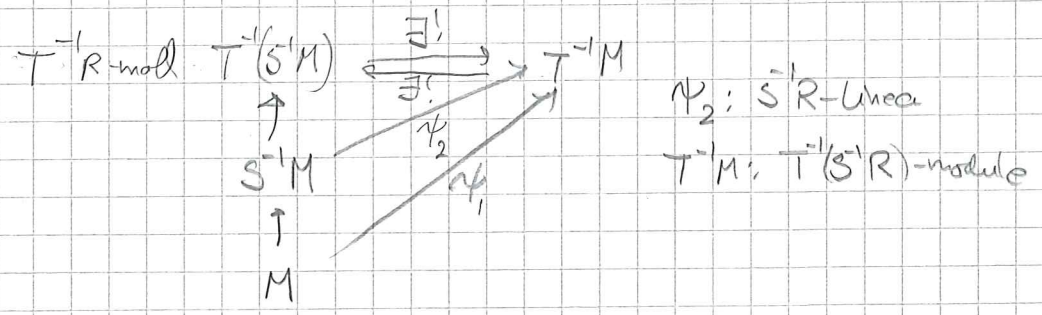
12.5
 $S \subseteq T \subseteq R$
 M
 R -module.

$$T^{-1}M = T^{-1}(S^{-1}M) = T_1^{-1}(S^{-1}M)$$

$$T_1 = \varphi_S(T) \subseteq S^{-1}R$$



$T^{-1}M$ has a $S^{-1}R$ -module structure since $S \subseteq T$, hence ψ factors through $S^{-1}M$



12.7
 $S \subseteq R$
 $m \in$

$$S^{-1}M = 0 \iff \text{Ann}(M) \cap S \neq \emptyset$$

$$\Rightarrow \text{if } M \text{ is finit.}$$

M
 R -module

pp $S^{-1}M = 0 \Rightarrow \forall m \in M \exists u \in S \text{ s.t. } um = 0$

finiten

$$M = \langle m_1, \dots, m_r \rangle, u_i m_i = 0$$

$$\sum \tau_i m_i = \sum \tau_i \prod_{j \neq i} u_j m_i = 0$$

$$\Rightarrow \prod u_i \in \text{Ann}(M) \text{ and } \prod u_i \in S$$

Let $x \in \text{Ann}(M) \cap S$, and $\frac{m}{s} \in S^{-1}M$.

Then $\frac{x}{1} \cdot \frac{m}{s} = 0$ i.e. $x \cdot \frac{m}{s} = 0$ or $\frac{x}{x} \cdot \frac{m}{s} = \frac{m}{s} = 0$
 since $x \in S$.

12.13

$S \subset R$
mc
 M, N
 R -modules

$$S^{-1}(M \otimes_R N) \stackrel{1)}{=} S^{-1}M \otimes_{S^{-1}R} N = S^{-1}M \otimes_{S^{-1}R} S^{-1}N = S^{-1}(M \otimes_R N)$$

$$\begin{aligned} \text{pp 1) } S^{-1}(M \otimes_R N) &\approx S^{-1}R \otimes_R (M \otimes_R N) \approx (S^{-1}R \otimes_R M) \otimes_R N \\ &\approx S^{-1}M \otimes_R N \end{aligned}$$

$$S^{-1}M \otimes_R N \approx S^{-1}M \otimes_{S^{-1}R} S^{-1}R \otimes_R N = S^{-1}M \otimes_{S^{-1}R} S^{-1}N$$

$$\text{ker } \varphi \rightarrow S^{-1}M \otimes_{S^{-1}R} S^{-1}N \rightarrow S^{-1}M \otimes_{S^{-1}R} S^{-1}N$$

gen by

$$\begin{aligned} \frac{xm}{s} \otimes \frac{n}{1} - \frac{m}{1} \otimes \frac{xn}{s} &= \frac{1}{s} (\frac{xm}{s} \otimes \frac{n}{1} - \frac{m}{1} \otimes \frac{xn}{s}) \\ &= \frac{1}{s} (xm \otimes n - m \otimes xn) = 0 \end{aligned}$$

12.24

$S = \mathbb{Z} - \{0\} \subset \mathbb{Z}$
 $M = \bigoplus_{n \geq 2} \mathbb{Z}/(n)$

$$\phi: S^{-1}\text{Hom}_{\mathbb{Z}}(M, M) \rightarrow \text{Hom}_{\mathbb{Q}}(S^{-1}M, S^{-1}M)$$

$$S^{-1}M = \bigoplus_{n \geq 2} S^{-1}\mathbb{Z}/(n)$$

$$S^{-1}\mathbb{Z}/(n) = 0 \iff \exists u \in S \text{ s.t. } u \cdot 1 = 0$$

We have $1 \in \text{Hom}_{\mathbb{Z}}(M, M)$ and $\phi_S(1) = 0$. In fact if $\phi_S(1) = 0$ then $\exists u \in S$ s.t. $u \cdot 1 = 0$. But $\forall u \in S$, we can find $n \geq 2$ s.t. $u \neq 0$ in $\mathbb{Z}/(n)$.