10.30 $R = k[x,y]/(y^2-x^2-x^3)$

Irreducible: $y^2-x^2-x^3 = f\cdot g$ $\quad f(0,0)=0 \Rightarrow f = f_1+f_2+f_3$

$\deg f, \deg g \geq 1$ $\quad g = g_0+g_1+g_2$

where $g_0 f_1 = g_2 f_3 = f_2 g_2 + f_3 g_1 = 0$

$\Rightarrow f$ or g is homogenous

$\Rightarrow y^2-x^2-x^3 = f_1(g_1+g_2)$

or

$= f_2(g_0+g_1)$

Both are impossible.

Not maximal $\quad (y^2-x^2-x^3) \subset (x+1, y-\sqrt{2})$

$$R = k[x,y]/(y^2-x^2-x^3) \xhookrightarrow{\text{injective}} k[t] \qquad \left(t = \frac{y}{x}\right)$$

$$x \longmapsto t^2-1$$

$$y \longmapsto t^3-t$$

Kernel: $(y^2-x^2-x^3) \longmapsto (t^3-t)^2-(t^2-1)^2-(t^2-1)^3 = 0.$

t integral $/R$ $\qquad p(t)=0$ where $p(z) = z^2-1-(t^2-1)$

$(\, p(t) = t^2 - t^2 = 0 \,)$

$\Downarrow$

$k[t]$ integral$/R$

$\frac{1}{t^2-1} \in \mathrm{Frac}(R) \Rightarrow \frac{t^3-t}{t^2-1} = t \in \mathrm{Frac}(R)$

$\Rightarrow \mathrm{Frac}(k[t]) \subset \mathrm{Frac}(R)$

$\Rightarrow \mathrm{Frac}(R) = \mathrm{Frac}(k[t])$

$$\begin{tikzcd}
R \arrow[r, hook] \arrow[d] & \overline{R} \arrow[r, hook] \arrow[d] & \mathrm{Frac}(R) \arrow[d, equal] \\
k[t] \arrow[r, equal] & k[t] \arrow[r, hook] & \mathrm{Frac}(k[t])
\end{tikzcd}$$

$\Rightarrow \boxed{\overline{R} = k[t].}$

$k[t]$ integral$/R$, $\mathrm{Frac}(R) = \mathrm{Frac}(k[t]) \Rightarrow k[t] \hookrightarrow \overline{R}.$

Lemma (technical)

$$\begin{tikzcd} R' \arrow[r, hook] & R'[X] \ni f = \underset{\text{monic}}{gh} \\ R \arrow[u, hook] \arrow[r, hook] & R[X] \ni \underset{\text{monic.}}{f} \arrow[u, hook] \end{tikzcd} \quad \Rightarrow \text{ coeff of } g \text{ and } h \text{ are integral}/R.$$

Pf

$$\begin{tikzcd} R' \arrow[r, hook] \arrow[rr, bend left] & R'[X] \arrow[r, two heads] & R'[X]/(g) \arrow[r, hook] & R'[x_1][X]. \end{tikzcd}$$

$$R'[X]/(g) = R'[x_1]$$

$g(X)$, $g(x_1) = 0$

$\Rightarrow g(X) = (X - x_1)\underset{\text{monic.}}{g_1(X)}$

Repeat: $g(X) = \prod_i (X - x_i)$ ..... $R' \hookrightarrow R'[x_1, .., x_n, y_1, .., y_m]$.

$h(X) = \prod_j (X - y_j)$

All x_i, y_j are roots of $f = gh$; hence integral$/R$.

$\Rightarrow$ coeff of g and h integral$/R$

Ex. $\mathbb{Z} \hookrightarrow \mathbb{Z}[i]$. $f(x) = x^2 + 1 = (x - i)(x + i) \in \mathbb{Z}[i][X]$

coeff are integral $/\mathbb{Z}$.

Theorem (Going down)

$$\begin{array}{ccc} & R' \supset \mathfrak{q}' & \\ \text{integral} & \uparrow \qquad \uparrow & \\ & R \supsetneq \mathfrak{q} \supsetneq \mathfrak{p} & \\ & \text{normal} \quad \overset{\parallel}{\mathfrak{q}'\cap R} \quad \text{primes} & \end{array} \Rightarrow \exists\ \mathfrak{p}' \subset \mathfrak{q}' \text{ s.t. } \mathfrak{p}' \cap R = \mathfrak{p}.$$

Pf. 1. $\mathfrak{p}\cdot R'_{\mathfrak{q}'} \cap R = \mathfrak{p}$. $\quad \mathfrak{p} \subset \mathfrak{p}R'_{\mathfrak{q}'} \cap R$ obvious.

Let $y = \frac{x}{s} \in \mathfrak{p}R'_{\mathfrak{q}'} \cap R$, i.e. $x \in \mathfrak{p}\cdot R'$, $s \in R' - \mathfrak{q}'$.

$\Downarrow$

$x = \sum_i y_i x_i \quad y_i \in \mathfrak{p} \quad x_i \in R'$

$$R \hookrightarrow \underset{R''}{\overset{\parallel}{R[x_1,\dots,x_n]}} \hookrightarrow R'$$

- finite R-module
- $x\cdot R'' = \sum y_i x_i R'' \in \mathfrak{p}\cdot R''$ i.e. $\mu_x : R'' \to R''$

Let $f(X)$ be the char. pol. of μ_x

$$f(X) = X^n + a_1 X^{n-1} + \dots + a_n \in R[X]$$

- $\mu_x(Z) \in \mathfrak{p}R'' \Rightarrow a_i \in \mathfrak{p}^i \subset \mathfrak{p}$.
- $f(x) = 0$. by Cayley-Hamilton

We can assume f is the minimal polynomial of $x\ /R$.

Pf. Suppose $f = gh$, $g, h \in K[X]$, monic $\quad (K = \mathrm{Frac}(R))$

R normal $\Rightarrow g, h \in R[X]$. $\quad f \equiv X^n\ (\mathfrak{p}) \Rightarrow g \equiv X^r\ (\mathfrak{p}) \quad h \equiv X^{n-r}\ (\mathfrak{p})$

since $(R/\mathfrak{p})[X]$ is a UFD.

$$\Rightarrow \quad s^n y^n + a_1 s^{n-1} y^{n-1} + \dots + a_n = y^n\left(s^n + \frac{a_1}{y} s^{n-1} + \dots + \frac{a_n}{y^n}\right) = 0$$

$x = s\cdot y$

coeff in K.

$$\left.\begin{array}{l} -\text{ minimal degree} \\ -\ s \text{ integral}/R \end{array}\right\} \Rightarrow \frac{a_i}{y^i} \in R.$$

If $y \notin \mathfrak{p}$, then $\frac{a_i}{y^i} \in \mathfrak{p}$ $\quad (y^i \cdot \frac{a_i}{y^i} \in \mathfrak{p})$

$\Rightarrow s^n \in \mathfrak{p}R' \subset \mathfrak{q}R' \subset \mathfrak{q}'$. But $s \in R' \setminus \mathfrak{q}'$. Contradiction

$\Rightarrow y \in \mathfrak{p} \Rightarrow \mathfrak{p}R'_{\mathfrak{q}'} \cap R \subset \mathfrak{p}$.

2. Remember.

$$\begin{array}{l} R' \\ \uparrow \varphi \\ R \supseteq \underset{\text{prime}}{\mathfrak{p}} \quad \text{s.t. } \varphi^{-1}(\mathfrak{p}R') = \mathfrak{p} \end{array} \qquad \Rightarrow \exists\, \mathfrak{q}' \subset R' \text{ s.t. } \varphi^{-1}(\mathfrak{q}') = \mathfrak{p}.$$

$\Rightarrow \exists\, \mathfrak{p}'' \subset R'_{\mathfrak{q}'}$ s.t. $\mathfrak{p}'' \cap R = \mathfrak{p}$.

Set $\mathfrak{p}' = \mathfrak{p}'' \cap R'$

$\Rightarrow \mathfrak{p}' \subset \mathfrak{q}'$ and $\mathfrak{p}' \cap R = \mathfrak{p}$.

$x \in \mathfrak{p} \subset R$
minimal prime

$\Downarrow$

x zero divisor

Pf. $R_{\mathfrak{p}} \supseteq \underset{\text{only prime}}{\mathfrak{p}R_{\mathfrak{p}}}$ Thus $y \in \mathfrak{p}R_{\mathfrak{p}} \Rightarrow y^n = 0$.

Now for $x \in \mathfrak{p}$ $\exists\, s \in R - \mathfrak{p}$ s.t. $sx^n = 0$ (n minimal)

$\Rightarrow x \cdot (sx^{n-1}) = 0$.