

10.30

$$R = k[x, y] / (y^2 - x^2 - x^3)$$

Irreducible: $y^2 - x^2 - x^3 = f \cdot g$
 $\deg f, \deg g \geq 1$

$$f(0,0) = 0 \Rightarrow f = f_1 + f_2 + f_3$$

$$g = g_0 + g_1 + g_2$$

$$\text{where } g_0 f_1 = g_2 f_3 = f_2 g_2 + f_3 g_1 = 0$$

$\Rightarrow f$ or g is homogeneous

$$\Rightarrow y^2 - x^2 - x^3 = f_1 (g_1 + g_2)$$

$$\text{or}$$

$$= f_2 (g_0 + g_1)$$

Both are impossible.

Not maximal $(y^2 - x^2 - x^3) \subset (x-1, y-\sqrt{2})$

$$R = k[x, y] / (y^2 - x^2 - x^3) \xrightarrow{\text{injective}} k[t] \quad (t = \frac{y}{x})$$

$$x \longmapsto t^2 - 1$$

$$y \longmapsto t^3 - t$$

$$\text{Kernel: } (y^2 - x^2 - x^3) \longmapsto (t^3 - t)^2 - (t^2 - 1)^2 - (t^2 - 1)^3 = 0.$$

$$t \text{ integral over } R \quad p(t) = 0 \text{ where } p(z) = z^2 - 1 - (z^2 - 1)$$

$$(p(t) = t^2 - t^2 = 0)$$



$k[t]$ integral over R

$$\frac{1}{t^2 - 1} \in R$$

$$\frac{1}{t^2 - 1} \in \text{Frac}(R) \Rightarrow \frac{t^3 - t}{t^2 - 1} = t \in \text{Frac}(R)$$

$$\Rightarrow \text{Frac}(k[t]) \subset \text{Frac}(R)$$

$$\Rightarrow \text{Frac}(R) = \text{Frac}(k[t]) = k(t)$$

$$R \hookrightarrow \bar{R} \hookrightarrow \text{Frac}(R)$$

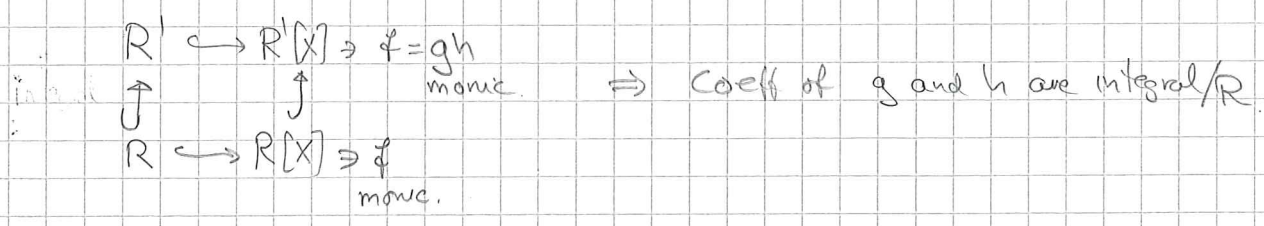
$$\downarrow \quad \downarrow \quad \parallel$$

$$k[t] = \bar{k}[t] \hookrightarrow \text{Frac}(k[t])$$

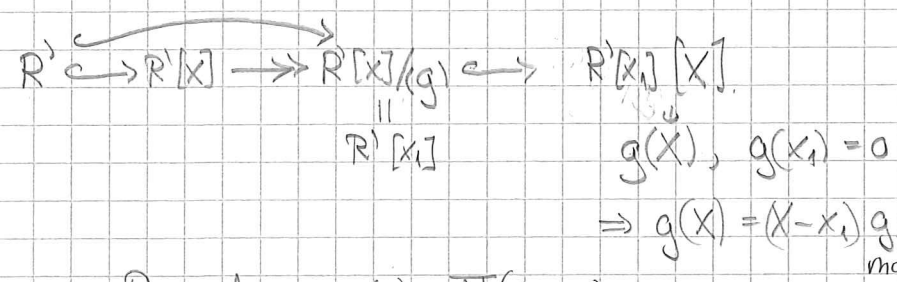
$$\bar{R} = k[t]$$

$$k[t] \text{ integral over } R, \text{Frac}(R) = \text{Frac}(k[t]) \Rightarrow k[t] \hookrightarrow \bar{R}$$

Lemma (technical)



Pf

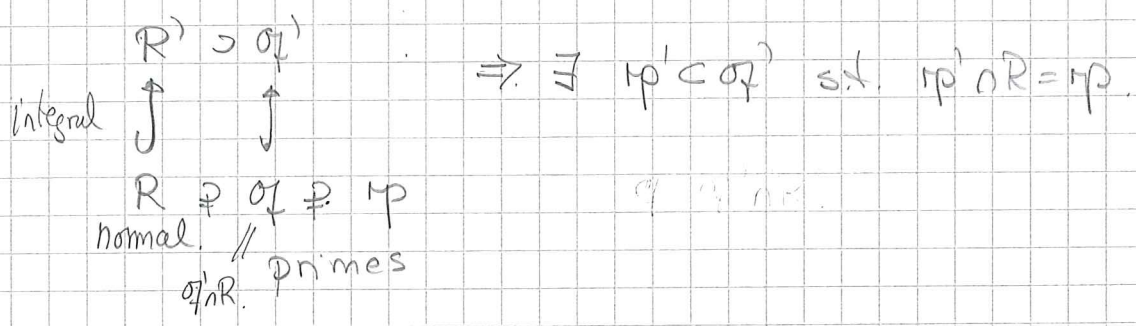


Repeat: $g(x) = \prod_i (x-x_i)$
 $h(x) = \prod_j (x-y_j)$ $\dots R' \hookrightarrow R'[x_1, x_2, \dots, x_n]$

All x_i, y_j are roots of $f = gh$ hence integral/ R .
 \rightarrow coeff of g and h integral/ R

Ex. $\mathbb{Z} \hookrightarrow \mathbb{Z}[i]. \quad f(x) = x^2 + 1 = (x-i)(x+i) \in \mathbb{Z}[i][X]$
 coeff are integral/ \mathbb{Z}

Theorem (Going down.)



Pf. 1. $\mathfrak{p} \cdot R_{\mathfrak{a}'} \cap R = \mathfrak{p}$. $\mathfrak{p} \subset \mathfrak{p} R_{\mathfrak{a}'} \cap R$ obvious.
 Let $y = \frac{x}{s} \in \mathfrak{p} R_{\mathfrak{a}'} \cap R$, i.e. $x \in \mathfrak{p} \cdot R'$, $s \in R' - \mathfrak{a}'$.
 $x = \sum_i y_i x_i$ $y_i \in \mathfrak{p}$ $x_i \in R'$

$$\begin{array}{c}
 R'' \\
 \parallel \\
 R \hookrightarrow R[x_1, \dots, x_n] \hookrightarrow R' \\
 \text{- finite } R\text{-module} \\
 \text{- } x \cdot R'' = \sum y_i x_i \in \mathfrak{p} \cdot R'' \text{ i.e. } \mu_x: R'' \rightarrow R''
 \end{array}$$

Let $f(x)$ be the char. pol. of μ_x
 $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0 \in R[x]$
 - $\mu_x(z) \in \mathfrak{p} \cdot R'' \Rightarrow a_i \in \mathfrak{p}^i \subset \mathfrak{p}$.
 - $f(x) = 0$. by Cayley-Hamilton

We can assume f is the minimal polynomial of x/R .

Pf. Suppose $f = gh$, $g, h \in K[x]$, monic ($K = \text{Frac}(R)$)
 R normal $\Rightarrow g, h \in R[x]$. $f \equiv x^n \pmod{\mathfrak{p}} \Rightarrow g \equiv x^r \pmod{\mathfrak{p}}$ $h \equiv x^{n-r} \pmod{\mathfrak{p}}$
 since $(R/\mathfrak{p})[x]$ is a UFD.

$$\begin{array}{l}
 \Rightarrow \\
 x = s \cdot y \\
 s^n y^n + a_1 s^{n-1} y^{n-1} + \dots + a_n = y^n \left(s^n + \frac{a_1}{y} s^{n-1} + \dots + \frac{a_n}{y^n} \right) = 0 \\
 \text{coeff in } K \\
 \left. \begin{array}{l} \text{- minimal degree} \\ \text{- } s \text{ integral}/R \end{array} \right\} \Rightarrow \frac{a_i}{y^i} \in R
 \end{array}$$

If $y \notin \mathfrak{p}$, then $\frac{a_i}{y_i} \in \mathfrak{p}$ ($y_i \cdot \frac{a_i}{y_i} \in \mathfrak{p}$)

$\Rightarrow s^n \in \mathfrak{p} \mathfrak{R}' \subset \mathfrak{q} \mathfrak{R}' \subset \mathfrak{q}'$. But $s \in \mathfrak{R}' - \mathfrak{q}'$ (contradiction)

$\Rightarrow y \in \mathfrak{p} \Rightarrow \mathfrak{p} \mathfrak{R}'_{\mathfrak{q}'} \cap \mathfrak{R} \subset \mathfrak{p}$.

2. Remember.

\mathfrak{R}'
 \uparrow
 $\mathfrak{R} = \mathfrak{p}$
 prime

$\Rightarrow \exists \mathfrak{q}' \subset \mathfrak{R}'$ s.t. $\mathfrak{p}'(\mathfrak{q}') = \mathfrak{p}$.

s.t. $\mathfrak{p}'(\mathfrak{p} \mathfrak{R}') = \mathfrak{p}$

$\Rightarrow \exists \mathfrak{p}'' \subset \mathfrak{R}'_{\mathfrak{q}'}$ s.t. $\mathfrak{p}'' \cap \mathfrak{R} = \mathfrak{p}$

Set $\mathfrak{p}' = \mathfrak{p}'' \cap \mathfrak{R}'$

$\Rightarrow \mathfrak{p}' \subset \mathfrak{q}'$ and $\mathfrak{p}' \cap \mathfrak{R} = \mathfrak{p}$.

$x \in \mathfrak{p} \subset \mathfrak{R}$
 minimal prime.

\Downarrow

x zero divisor

Pf.

$\mathfrak{R}_{\mathfrak{p}} = \mathfrak{p} \mathfrak{R}_{\mathfrak{p}}$
 only prime

thus $y \in \mathfrak{p} \mathfrak{R}_{\mathfrak{p}} \Rightarrow y^n = 0$.

Now for $x \in \mathfrak{p} \exists s \in \mathfrak{R} - \mathfrak{p}$ s.t. $sx^n = 0$ (n minimal)

$\Rightarrow x \cdot (sx^{n-1}) = 0$