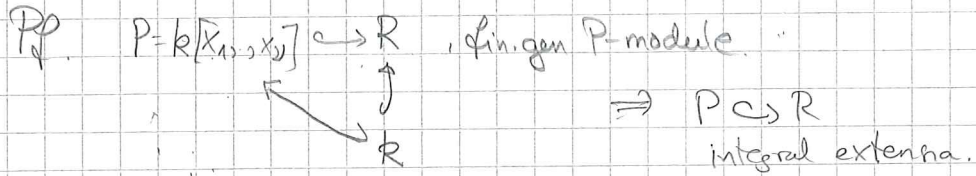
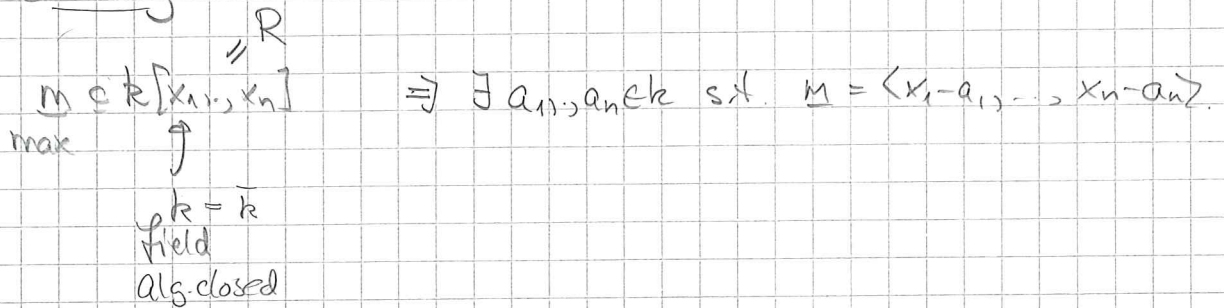


Weak Nullstellensatz.



$$\text{Field } R \Rightarrow \text{Field } P \Rightarrow \mathcal{V} = \emptyset \Rightarrow P = k$$

Corollary



Pf. Let $K = R/\mathfrak{m} \cong k \Rightarrow k \hookrightarrow K \Rightarrow K = k$
 (by $k = \bar{k}$)

Let $R \rightarrow R/\mathfrak{m} \cong k$
 $x_i \mapsto a_i$

$$\Rightarrow \langle x_1 - a_1, \dots, x_n - a_n \rangle \subset \mathfrak{m}$$

On the other hand

$$\begin{array}{ccc}
 R[X_1, \dots, X_n] & \xrightarrow{\varphi} & R \\
 X_i & \mapsto & x_i
 \end{array}$$

$$\varphi(\langle x_1 - a_1, \dots, x_n - a_n \rangle) = \langle x_1 - a_1, \dots, x_n - a_n \rangle$$

\uparrow
 maximal in the k -alg \Rightarrow maximal $\Rightarrow \langle x_1 - a_1, \dots, x_n - a_n \rangle = \mathfrak{m}$

Hilbert Nullstellensatz

$$\begin{array}{l}
 R \\
 \text{field}
 \end{array}
 \alpha \notin R = k[x_1, \dots, x_n]
 \begin{array}{l}
 \text{fin. gen. } k\text{-alg.}
 \end{array}
 \Rightarrow \sqrt{(\alpha)} = \bigcap_{\substack{M \\ \text{max.}}} M$$

PP 1. Replace R by R/α , i.e. assume $\alpha = (0)$

2. $\sqrt{(0)} \subset \bigcap_M M$ obvious

3. $f \notin \sqrt{(0)}$. $\Rightarrow R_f \neq 0$. $\bigcap_{\text{max}} n \subset R_f \xleftarrow{\varphi_f} R$

Put $m = \varphi_f^{-1}(n)$

R fin. gen. $\Rightarrow R_f = R[X]/(X^q - 1)$ fin. gen.

$\Rightarrow R_f/m$ fin. field ext. of k (weak nullst. satz)

4. $k \hookrightarrow R/m \hookrightarrow R_f/n$
 $\begin{array}{l} \text{fin. gen.} \\ k\text{-module} \end{array} \longleftarrow \text{fin. ext}$

$\Rightarrow k \hookrightarrow R/m$ integral extension

k field $\Rightarrow R/m$ field $\Rightarrow m$ maximal ideal

5. $\frac{f}{1} \in R_f \Rightarrow \frac{f}{1} \in n \Rightarrow f \notin m \Rightarrow f \notin \bigcap_M M$
 unit

$$\Rightarrow \bigcap_M M \subset \sqrt{(0)}$$

Noether Normalization

Lemma 1. Let $f \in k[x_1, \dots, x_n]$, $n \geq 2$, be a non-zero polynomial over an infinite field k . Then there are elements $\lambda, a_1, \dots, a_{n-1} \in k$ such that the polynomial

$$\lambda f(y_1 + a_1 y_n, \dots, y_{n-1} + a_{n-1} y_n, y_n) \in k[y_1, \dots, y_n]$$

is monic in y_n .

Proof. Let f_d be the homogenous part of f of highest degree. We have

$$f_d(\lambda x_1, \dots, \lambda x_n) = \lambda^d f(x_1, \dots, x_n)$$

where d is the degree of f . Since k is infinite we can always find a_1, \dots, a_{n-1} such that $f_d(a_1, \dots, a_{n-1}, 1) \neq 0$. Let

$$y_j = x_j - a_j x_n, \quad j = 1, 2, \dots, n-1 \quad \text{and} \quad y_n = x_n$$

and $\lambda = f_d(a_1, \dots, a_{n-1}, 1)^{-1}$. Then

$$\begin{aligned} & \lambda f(y_1 + a_1 y_n, \dots, y_{n-1} + a_{n-1} y_n, y_n) \\ &= \lambda f_d(a_1, \dots, a_{n-1}, 1) y_n^d + \text{terms of lower degree in } y_n \end{aligned}$$

Which is a monic polynomial in y_n . □

Theorem 1 (Noether Normalization). Let R be a finitely generated algebra over a infinite field k , with generators $x_1, \dots, x_n \in R$. Then there is an injective k -algebra homomorphism $\phi : k[t_1, \dots, t_r] \rightarrow R$ from a polynomial ring to R , such that R is integral over $k[t_1, \dots, t_r]$.

Example 1. Let $R = k[x_1, x_2]/(x_1 x_2 - 1) \cong k[x, \frac{1}{x}]$. Then R is not integral over $k[x]$. In fact, suppose $\frac{1}{x}$ is integral over $k[x]$. Then there would exist a polynomial equation

$$\left(\frac{1}{x}\right)^n + a_1(x) \left(\frac{1}{x}\right)^{n-1} + \dots + a_n(x) = 0$$

where $a_i(x) \in k[x]$. Multiplying by x^{n-1} now gives

$$\frac{1}{x} = -a_1(x) - a_2(x)x - \dots - a_n(x)x^{n-1} \in k[x]$$

and $\frac{1}{x} \in k[x]$, which is not true.

If we introduce new coordinates, $x_1 = t_1 + t_2$, $x_2 = t_2$, we can write $R = k[t_1, t_2]/(t_2^2 + t_1 t_2 - 1)$. Then there is an injective map $\phi : k[t_1] \rightarrow R$, and R is integral over $k[t_1]$.

Proof. We prove the statement by induction on the number n of generators of R . If $n = 1$ we let $t_1 = x_1$.

Assume $n > 1$. If the generators x_1, \dots, x_n are algebraically independent, we choose $t_i = x_i$, and the result follows.

Suppose there is an algebraic dependence between the generators, i.e. a non-zero polynomial f over k such that $f(x_1, \dots, x_n) = 0$. Let f_d be the homogenous part of the highest degree of f . By lemma 1 we can find a_1, \dots, a_{n-1} such that

$$\lambda f(y_1 + a_1 y_n, \dots, y_{n-1} + a_{n-1} y_n, y_n) \in k[y_1, \dots, y_n]$$

is monic in y_n . The new coordinates are given by

$$y_1 = x_1 - a_1 x_n, \dots, y_{n-1} = x_{n-1} - a_{n-1} x_n, y_n = x_n$$

and it follows that

$$\lambda f(y_1 + a_1 y_n, \dots, y_{n-1} + a_{n-1} y_n, y_n) = \lambda f(x_1, \dots, x_n) = 0$$

Notice that the subalgebra $k[y_1, \dots, y_n]$ of R , generated by y_1, \dots, y_n is the same as generated by x_1, \dots, x_n , i.e. all of R . Moreover, y_n is integral over the subalgebra

$k[y_1, \dots, y_{n-1}]$ by the result of lemma 1. By the induction hypothesis there is a injective algebra homomorphism $\phi : k[t_1, \dots, t_r] \rightarrow k[y_1, \dots, y_{n-1}]$, such that $k[y_1, \dots, y_{n-1}]$ is integral over $k[t_1, \dots, t_r]$. But y_n is integral over $k[y_1, \dots, y_{n-1}]$, and by the Tower Law of integrality, y_n is integral over $k[t_1, \dots, t_r]$, and the result follows. \square

Let us see how the proof works for the above example.

Example 2. Let $R = k[x_1, x_2]/(x_1x_2 - 1)$. Then $f(x_1, x_2) = x_1x_2 - 1 = 0$, and the homogenous part of highest degree is $f_2(x_1, x_2) = x_1x_2$. We have $\lambda^{-1} = f_2(1, 1) = 1$. Let $y_1 = x_1 - x_2$, $y_2 = x_2$. Then

$$\lambda f(y_1 + y_2, y_2) = (y_1 + y_2)y_2 - 1 = y_2^2 + y_1y_2 - 1 = x_1x_2 - 1 = 0$$

and y_2 is integral over $k[y_1]$. Let $t_1 = y_1$. Then $k[t_1] \hookrightarrow k[y_1, y_2] = k[x_1, x_2]$ is an integral extension.