Weak Nullstellensatz.

k field, R fin. gen k-alg. R field $\Rightarrow$ R fin. ext. field of k.

Pf. $P = k[X_1, \dots, X_\nu] \hookrightarrow R$, fin. gen P-module

$$\begin{tikzcd} P = k[X_1,\dots,X_\nu] \arrow[r, hook] & R \\ & k \arrow[u, hook] \arrow[ul] \end{tikzcd}$$

$\Rightarrow$ $P \hookrightarrow R$ integral extension.

R field $\Rightarrow$ P field $\Rightarrow \nu = 0 \Rightarrow P = k$.

Corollary

$\underline{m} \subseteq k[x_1, \dots, x_n] = R$, $\underline{m}$ max, $k = \bar{k} \hookrightarrow k[x_1,\dots,x_n]$ field alg. closed $\Rightarrow \exists\, a_1, \dots, a_n \in k$ s.t. $\underline{m} = \langle x_1 - a_1, \dots, x_n - a_n \rangle$.

Pf. Let $K = R/\underline{m}$ (fin. gen$/k$) $\hookleftarrow k$ $\Rightarrow$ $k \hookrightarrow K$ fin. ext. $\Rightarrow$ $K = k$ (by $k = \bar{k}$)

Let. $R \longrightarrow R/\underline{m} \simeq k$

$x_i \longmapsto a_i$

$\Rightarrow$ $(x_1 - a_1, \dots, x_n - a_n) \subset \underline{m}$.

On the other hand

$$k[X_1, \dots, X_n] \xrightarrow{\varphi} R$$

$$X_i \mapsto x_i$$

$$\varphi(X_1 - a_1, \dots, X_n - a_n) = (x_1 - a_1, \dots, x_n - a_n)$$

$(X_1 - a_1, \dots, X_n - a_n)$ maximal in the k-alg $\Rightarrow$ maximal $\Rightarrow (x_1 - a_1, \dots, x_n - a_n) = \underline{m}$

Hilbert Nullstellensatz

k field, $\mathfrak{a} \subsetneq R = k[x_1, .., x_n]$ fin. gen. k-alg. $\Longrightarrow$ $\sqrt{\mathfrak{a}} = \bigcap\limits_{\substack{\underline{m} \supset \mathfrak{a} \\ \text{max}}} \underline{m}$

Pf. 1. Replace R by $R/\mathfrak{a}$, i.e. assume $\mathfrak{a} = (0)$

2. $\sqrt{(0)} \subset \bigcap\limits_{\underline{m}} \underline{m}$ obvious

3. $f \notin \sqrt{(0)}$. $\Rightarrow R_f \neq 0$. $\underset{\text{max}}{\underline{n}} \subset R_f \xleftarrow[\varphi_f]{} R$

Put $\underline{m} = \varphi_f^{-1}(\underline{n})$.

$\underset{\text{fin. gen}}{R} \Rightarrow \underset{\text{fin gen}}{R_f = R[X]/(Xf-1)}$

$\Rightarrow R_f/\underline{n}$ fin. field ext. of k (weak nullst.satz)

4. $k \hookrightarrow R/\underline{m} \hookrightarrow R_f/\underline{n}$

fin. gen. k-module $\Longleftarrow$ fin. ext

$\Rightarrow k \hookrightarrow R/\underline{m}$ integral extension

k field $\Rightarrow$ $R/\underline{m}$ field $\Rightarrow$ $\underline{m}$ maximal ideal

5. $\underset{\text{unit}}{\frac{f}{1} \in R_f} \Rightarrow \frac{f}{1} \notin \underline{n} \Rightarrow f \notin \underline{m} \Rightarrow f \notin \bigcap\limits_{\underline{m}} \underline{m}$

$\Longrightarrow \bigcap\limits_{\underline{m}} \underline{m} \subset \sqrt{(0)}$

Noether Normalization

Lemma 1. *Let $f \in k[x_1, \dots, x_n]$, $n \geq 2$, be a non-zero polynomial over an infinite field k. Then there are elements $\lambda, a_1, \dots, a_{n-1} \in k$ such that the polynomial*

$$\lambda f(y_1 + a_1 y_n, \dots, y_{n-1} + a_{n-1} y_n, y_n) \in k[y_1, \dots, y_n]$$

is monic in y_n.

Proof. Let f_d be the homogenous part of f of highest degree. We have

$$f_d(\lambda x_1, \dots, \lambda x_n) = \lambda^d f(x_1, \dots, x_n)$$

where d is the degree of f. Since k is infinite we can always find $a_1, \dots, a_{n-1}$ such that $f_d(a_1, \dots, a_{n-1}, 1) \neq 0$. Let

$$y_j = x_j - a_j x_n, \quad j = 1, 2, \dots, n-1 \quad \text{and} \quad y_n = x_n$$

and $\lambda = f_d(a_1, \dots, a_{n-1}, 1)^{-1}$. Then

$$\begin{aligned} &\lambda f(y_1 + a_1 y_n, \dots, y_{n-1} + a_{n-1} y_n, y_n) \\ &\quad = \lambda f_d(a_1, \dots, a_{n-1}, 1) y_n^d + \text{terms of lower degree in } y_n \end{aligned}$$

Which is a monic polynomial in y_n. □

Theorem 1 (Noether Normalization). *Let R be a finitely generated algebra over a infinite field k, with generators $x_1, \dots, x_n \in R$. Then there is an injective k-algebra homomorphism $\phi : k[t_1, \dots, t_r] \to R$ from a polynomial ring to R, such that R is integral over $k[t_1, \dots, t_r]$.*

Example 1. *Let $R = k[x_1, x_2]/(x_1 x_2 - 1) \cong k[x, \frac{1}{x}]$. Then R is not integral over $k[x]$. In fact, suppose $\frac{1}{x}$ is integral over $k[x]$. Then there would exist a polynomial equation*

$$(\frac{1}{x})^n + a_1(x)(\frac{1}{x})^{n-1} + \dots + a_n(x) = 0$$

where $a_i(x) \in k[x]$. Multiplying by x^{n-1} now gives

$$\frac{1}{x} = -a_1(x) - a_2(x)x - \dots - a_n(x)x^{n-1} \in k[x]$$

and $\frac{1}{x} \in k[x]$, which is not true.

If we introduce new coordinates, $x_1 = t_1 + t_2$, $x_2 = t_2$, we can write $R = k[t_1, t_2]/(t_2^2 + t_1 t_2 - 1)$. Then there is an injective map $\phi : k[t_1] \to R$, and R is integral over $k[t_1]$.

Proof. We prove the statement by induction on the number n of generators of R. If $n = 1$ we let $t_1 = x_1$.

Assume $n > 1$. If the generators $x_1, \dots, x_n$ are algebraically independent, we choose $t_i = x_i$, and the result follows.

Suppose there is an algebraic dependence between the generators, i.e. a non-zero polynomial f over k such that $f(x_1, \dots, x_n) = 0$. Let f_d be the homogenous part of the highest degree of f. By lemma 1 we can find $a_1, \dots, a_{n-1}$ such that

$$\lambda f(y_1 + a_1 y_n, \dots, y_{n-1} + a_{n-1} y_n, y_n) \in k[y_1, \dots, y_n]$$

is monic in y_n. The new coordinates are given by

$$y_1 = x_1 - a_1 x_n, \ \dots, y_{n-1} = x_{n-1} - a_{n-1} x_n, y_n = x_n$$

and it follows that

$$\lambda f(y_1 + a_1 y_n, \dots, y_{n-1} + a_{n-1} y_n, y_n) = \lambda f(x_1, \dots, x_n) = 0$$

Notice that the subalgebra $k[y_1, \dots, y_n]$ of R, generated by $y_1, \dots, y_n$ is the same as generated by $x_1, \dots, x_n$, i.e. all of R. Moreover, y_n is integral over the subalgebra

$k[y_1, \ldots, y_{n-1}]$ by the result of lemma 1. By the induction hypothesis there is a injective algebra homomorphism $\phi : k[t_1, \ldots, t_r] \to k[y_1, \ldots, y_{n-1}]$, such that $k[y_1, \ldots, y_{n-1}]$ is integral over $k[t_1, \ldots, t_r]$. But y_n is integral over $k[y_1, \ldots, y_{n-1}]$, and by the Tower Law of integrality, y_n is integral over $k[t_1, \ldots, t_r]$, and the result follows. □

Let us see how the proof works for the above example.

Example 2. *Let $R = k[x_1, x_2]/(x_1x_2 - 1)$. Then $f(x_1, x_2) = x_1x_2 - 1 = 0$, and the homogenous part of highest degree is $f_2(x_1, x_2) = x_1x_2$. We have $\lambda^{-1} = f_2(1, 1) = 1$. Let $y_1 = x_1 - x_2$, $y_2 = x_2$. Then*

$$\lambda f(y_1 + y_2, y_2) = (y_1 + y_2)y_2 - 1 = y_2^2 + y_1y_2 - 1 = x_1x_2 - 1 = 0$$

and y_2 is integral over $k[y_1]$. Let $t_1 = y_1$. Then $k[t_1] \hookrightarrow k[y_1, y_2] = k[x_1, x_2]$ is an integral extension.