Def A ring R is Noetherian if every ideal is fin. gen.

Ex. $\underset{\text{PID}}{R} \Rightarrow$ Noetherian $(k, k[X], \mathbb{Z})$

$k[X_1, \ldots, X_n]$ Noetherian

$k[X_1, \ldots, \ldots]$ non-Noetherian

$R = \{a + X \cdot g(X,Y) \mid a \in k,\ g(X,Y) \in k[X,Y]\} \subset k[X,Y]$

$\mathfrak{a} = \langle X, XY, XY^2, \ldots \rangle$ not fin. gen

$R = k[X, Y, \frac{X}{Y}, \frac{X}{Y^2}, \ldots] \subset k(X,Y)$ $\quad (X) \subsetneq (\frac{X}{Y}) \subsetneq (\frac{X}{Y^2}) \subsetneq \cdots$

(acc): Every ascending chain $\mathfrak{a}_0 \subsetneq \mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots$ stabilizes
ascending chain cond. i.e. $\exists N$ s.t. if $j \geq N$, $\mathfrak{a}_j = \mathfrak{a}_N$.

(maxc): If every non-empty set S of ideals
Maximal condition contains maximal elements

Noetherian $\overset{3}{\Rightarrow}$ Acc $\overset{1}{\Leftrightarrow}$ maxc $\overset{2}{\Rightarrow}$ Noetherian

Pf $\overset{1}{\Leftarrow}$ $S = \{\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots\}$ contains max. $\mathfrak{a}_N \Rightarrow \mathfrak{a}_j = \mathfrak{a}_N$ $j \geq N$

$\overset{1}{\Rightarrow}$ Assume $S = \{\mathfrak{a}_i\}_{i \in I}$ has no maximal element. Then for $\mathfrak{a}_0 \in S$ we can find $\mathfrak{a}_0 \subsetneq \mathfrak{a}_1$, and for $\mathfrak{a}_1$ we can find $\mathfrak{a}_0 \subsetneq \mathfrak{a}_1 \subsetneq \mathfrak{a}_2$. By axiom of countable choice we can find $\mathfrak{a}_0 \subsetneq \mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \cdots\cdots$ (Infinite).

$\overset{3}{\Rightarrow}$ $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$. Let $\mathfrak{a} = \bigcup \mathfrak{a}_i$ ideal, fin. gen; $\mathfrak{a} = (f_1, \ldots, f_m)$
Then $f_i \in \mathfrak{a}_{j_i}$ and let $N = \max(j_i) \Rightarrow \mathfrak{a}_j = \mathfrak{a}_N$ for $j \geq N$

$\overset{2}{\Rightarrow}$ $\mathfrak{a} = \langle f_\lambda ; \lambda \in \Lambda \rangle$ (possible infinite). S: set of all ideals gen by fin. many f_λ

$\Rightarrow S$ has max element; $\mathfrak{b} \in S$, $\mathfrak{b} = \langle f_{\lambda_1}, \ldots, f_{\lambda_r} \rangle$.

We have $\mathfrak{b} \subseteq \mathfrak{b} + (f_\lambda)$, $f_\lambda \in \mathfrak{b}$. By maxc $\mathfrak{b} = \mathfrak{b} + (f_\lambda) \Rightarrow f_\lambda \in \mathfrak{b}$

$\Rightarrow \mathfrak{b} \subseteq \mathfrak{a} \subseteq \mathfrak{b} \Rightarrow \mathfrak{a} = \mathfrak{b}$ and $\mathfrak{a}$ is fin. gen

Prop $\underset{\text{Noetherian ring}}{\overset{\text{-mc}}{S} \subset R \supset \overset{\text{ideal}}{\mathfrak{a}}} \Rightarrow \underset{\text{Noetherian}}{R/\mathfrak{a}, \; S^{-1}R}$

Pf R satisfies acc on ideals

$\Rightarrow R/\mathfrak{a}, S^{-1}R$ satisfy acc on ideals

Theorem (Cohen)

A ring R is Noetherian if every prime ideal is fin gen

Pf Let S = set of non-fin.gen. ideals. Suppose $S \neq \emptyset$. Let $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \dots \in S$. Let $\mathfrak{a} = \bigcup \mathfrak{a}_i$. If $\mathfrak{a}$ is fin.gen, $\mathfrak{a} = (f_1, \dots, f_r)$ Let i_j be the smallest number such that $f_j \in \mathfrak{a}_{i_j}$, and let $N = \max(i_j)$ Then $\mathfrak{a} = \mathfrak{a}_N$ and $\mathfrak{a}_N \in S$ is fin.gen contradiction $\Rightarrow \mathfrak{a} \in S$

Zorns Lemma $\Rightarrow \exists \, \mathfrak{p} \in S$ max element, non fin.gen

Suppose every prime is fin.gen. $\Rightarrow \mathfrak{p}$ is not a prime

$\Rightarrow \exists a, b \in R \setminus \mathfrak{p}$ s.t. $ab \in \mathfrak{p}$

$$\mathfrak{p} \subsetneq \mathfrak{p} + (a) \Rightarrow \mathfrak{p} + (a) \text{ is fin.gen}$$
$$\mathfrak{p} + (a) = (x_1 + t_1 a, \dots, x_n + t_n a, a) = (x_1, \dots, x_n, a)$$

Let $\mathfrak{b} = \mathrm{Ann}\left(\mathfrak{p}+(a)/\mathfrak{p}\right) \Rightarrow \mathfrak{p} + (b) \subset \mathfrak{b}, \; b \notin \mathfrak{p}$

$\Rightarrow \mathfrak{p} \subsetneq \mathfrak{b} \Rightarrow \mathfrak{b}$ fin.gen, $\mathfrak{b} = \langle y_1, \dots, y_m \rangle$

Let $z \in \mathfrak{p} \Rightarrow z \in \mathfrak{p} + (a)$, i.e. $z = \underbrace{a_1 x_1 + \dots + a_n x_n}_{\in \mathfrak{p}} + y a \Rightarrow ya \in \mathfrak{p}$

$\Rightarrow y(\mathfrak{p} + (a)) = y \cdot \mathfrak{p} + (ya) \in \mathfrak{p}$

$\Rightarrow y \in \mathfrak{b}$ i.e. $y = b_1 y_1 + \dots + b_m y_m$

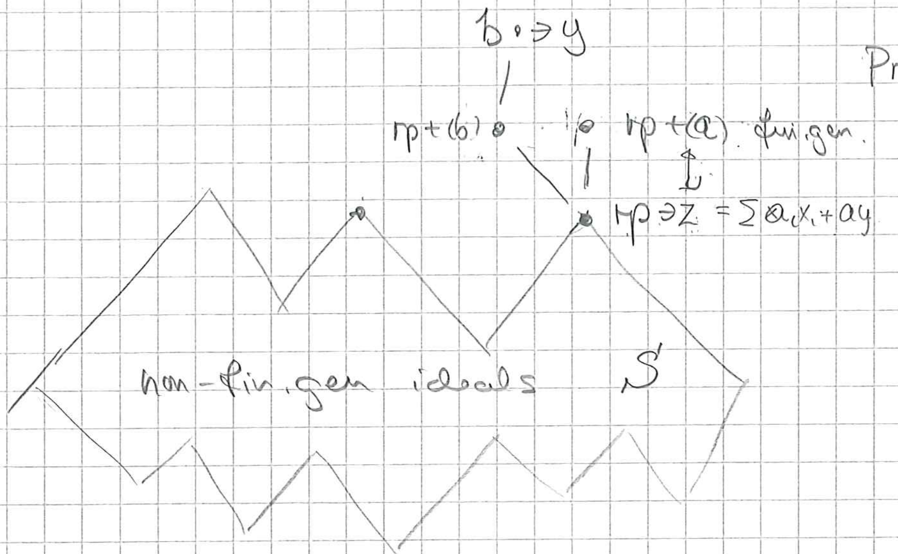
$\Rightarrow \mathfrak{p} = \langle x_1, \dots, x_n, a y_1, \dots, a y_m \rangle$ contradiction (fin.gen)

$\Rightarrow R$ Noetherian

Primes are fin. gen

$$\mathfrak{b} = \mathrm{Ann}\left(\left(\mathfrak{p}+(a)\right)/\mathfrak{p}\right)$$
$$\shortparallel$$
$$(\mathfrak{p}+(a) : \mathfrak{p})$$

Ul :



$$R \Longrightarrow R[X]$$
Noeth. Noeth

Pf. $\mathfrak{a} \subset R[X]$
non. fin. gen

$\mathfrak{a}_0 = (0)$

Inductively $f_i \in \mathfrak{a} - \mathfrak{a}_{i-1}$ minimal degree d_i

$\rightsquigarrow \mathfrak{a}_i = \mathfrak{a}_{i-1} + (f_i)$ $\quad d_i \leq d_{i+1}$

$f_i = a_i X^{n_i} +$ lower degree terms.

$\mathfrak{b} = (a_1, \ldots \quad) \subset R \Rightarrow$ fin. gen.

$\Rightarrow \mathfrak{b} = (a_1, \ldots, a_n)$.

Put $a_{n+1} = r_1 a_1 + \cdots + r_n a_n \quad r_j \in R$.

Set $f := f_{n+1} - \left(r_1 f_1 X^{d_{n+1}-d_1} + \cdots + r_n f_n X^{d_{n+1}-d_n}\right)$

$\Longrightarrow \deg f < d_{n+1} \Rightarrow f \in \mathfrak{a}_n \Rightarrow f_{n+1} \in \mathfrak{a}_n$.

Contradiction

Theorem (Hilbert basis)

$\underset{\text{Noeth.}}{R}$ $\quad \underset{\substack{\text{fin. gen} \\ R\text{-alg.}}}{R'}$ $\quad \Rightarrow \underset{\text{Noeth.}}{R'}$

Pf. $R' \twoheadleftarrow \underset{\text{Noeth.}}{R[x_1, \dots, x_n]}$

DCC on ideals. $\Rightarrow$ Artinian rings.

<u>Opposite:</u> DCC (Descending chain condition)

$\dots \subset \mathfrak{a}_1 \subset \mathfrak{a}_0$ stabilizes i.e. $\mathfrak{a}_i = \mathfrak{a}_{i+1}$, $i > N$

$\underset{\text{field}}{k}$ $\quad \underset{\substack{\text{fin. gen} \\ k\text{-alg.}}}{A}$ $\qquad \underset{\text{Artinia}}{A} \iff \dim_k A < \infty.$

In an Artinian ring every prime ideal is max.

Pf. $\underset{\text{prime}}{\mathfrak{p} \subset A} \Rightarrow \underset{\text{Artinian domain}}{A/\mathfrak{p}} = B \ni x, \ x \neq 0$

$\text{DCC} \Rightarrow (x^{n+1}) = (x^n)$ for some n

$\Rightarrow x^n = y \cdot x^{n+1}, \ y \in B$

or $x^n(1 - xy) = 0$

$\underset{\text{domain}}{B} \Rightarrow 1 - xy = 0$ or $xy = 1$.

$\Rightarrow x$ has invers in B.

$\Rightarrow B$ is field

$\Rightarrow \mathfrak{p} \subset A$ is max.