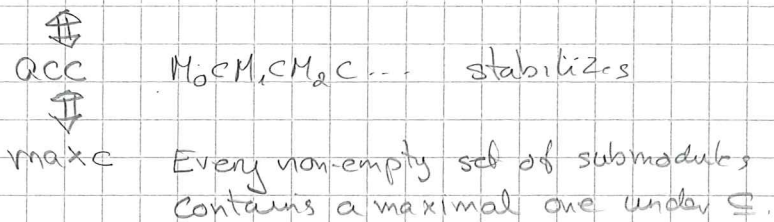


Def
 R ring
 M R -module
 Noetherian

if every submodule is fin. gen.



(Usefull lemma)

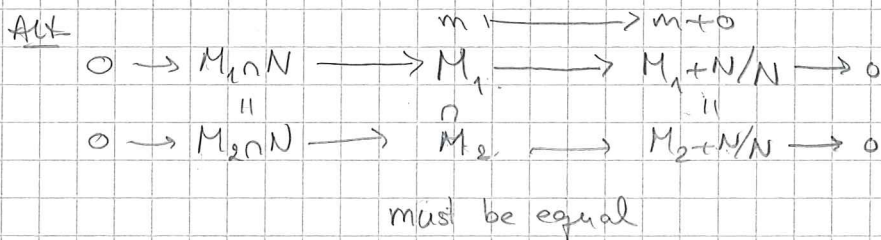
R ring, $M \supset M_2 \supset M_1$ R -module. If $\exists N \subset M$ such that

- 1) $M_1 \cap N = M_2 \cap N$
- 2) $M_1 + N / N = M_2 + N / N$

Then $M_1 = M_2$

Doesn't need

Pf
 Let $m_2 \in M_2$. Then ^{by 2)} modulo $N \exists m_1 \in M_1$ and $n \in N$ s.t. $m_2 \equiv m_1 + n \pmod{N}$, i.e. $n = m_2 - m_1 \in N$. But $M_1 \subset M_2$, thus $m_1 \in M_2$ and $n = m_2 - m_1 \in M_2$ i.e. $n = m_2 - m_1 \in M_2 \cap N = M_1 \cap N \Rightarrow m_2 = n + m_1 \in M_1$ \square

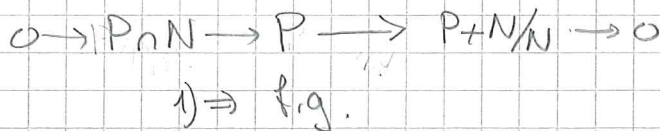


Proposition.

- R ring, $M \supset N$ R -module
- 1) M fin. gen. $\iff N, M/N$ fin. gen.
 - 2) M Noetherian $\iff N, M/N$ Noetherian.

Pf. $M \Rightarrow$ Every submod. of N is f.g. $\varphi: M \rightarrow M/N \supset P$
 $\varphi(P)$ fin. gen. $\Rightarrow P$ fin. gen.

$N, M/N, P \subset M$. $P \cap N, P + N / N$.
 Noeth. fin. gen. by assumption



$$M_1, \dots, M_n \text{ Noeth.} \iff M_1 \oplus \dots \oplus M_n \text{ Noetherian.}$$

Pf. For $n=2$.

$$0 \rightarrow M_1 \rightarrow M_1 \oplus M_2 \rightarrow M_2 \rightarrow 0$$

$$\quad \quad \quad \parallel \quad \quad \quad \parallel$$

$$\quad \quad \quad M_1 \quad \quad \quad M/M_1$$

Follows from Prop.

Artinian algebras:

(A, \mathfrak{m})
local
Artinian
f.g. k -algebra.

By Weak Nullstellensatz or Zariskus Lemma

$$k \xrightarrow{\quad} A \xrightarrow{\quad} A/\mathfrak{m}$$

field
finite alg. ext. of k

A local Artinian $\Rightarrow \mathfrak{m}^n = 0$ for some n

$$\Rightarrow A \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \dots \supset \mathfrak{m}^n = 0$$

A Noeth. $\Rightarrow \mathfrak{m}^i/\mathfrak{m}^{i+1}$ f.g. A/\mathfrak{m} -module

$$\Rightarrow A \text{ finite dimensional } /k.$$

$\dim_k A = 1.$

$A \cong k.$

$\dim_k A = 2$

$A \cong k[X]/(X^2+ax+b) \quad \Delta = a^2-4b.$

$\Delta = 0$: one root : $k[X]/(X^2)$

$\Delta \neq 0$: two roots : $k[X]/(X^2-1) \cong k \oplus k \quad 1 \mapsto (1,1), X \mapsto (1,-1)$

$\dim_k A = 3.$

$A = \langle 1, e_2, e_3 \rangle$ as k -vector space

$e_2^2 = a_1 + a_2 e_2 + a_3 e_3$ By coordinate change

1) $e_2^2 = e_3$ 2) $e_2^2 = 1$ 3) $e_2^2 = 0$

By $e_2 e_3 = b_1 + b_2 e_2 + b_3 e_3$

1) $e_2^3 - b_3 e_2^2 - b_2 e_2 - b_1 = 0 \quad k[X]/(X^3 - b_3 X^2 - b_2 X - b_1)$

2) $e_2^2 = 1 \Rightarrow (b_3^2 - 1)e_3 + (b_1 + b_2 b_3)e_2 + (b_2 + b_1 b_3) = 0$

If $b_3^2 \neq 1$, then $e_3 \in \langle 1, e_2 \rangle$ and $\dim_k A = 2$. (Can assume $b_3 = 1$)

$$\Rightarrow (b_1 + b_2)e_2 + (b_1 + b_2)1 = 0 \Rightarrow b_1 = -b_2$$

$$\Rightarrow e_2 e_3 = b - b e_2 + e_3 \quad \text{or} \quad (e_3 + b)e_2 = e_3 + b.$$

By coord. change $(e_2, e_3) = (e_2, e_3)$

$$\text{Put } e_3^2 = c_1 + c_2 e_2 + c_3 e_3 \quad \text{and} \quad (e_2 e_3 - e_3)^2 = e_3^2 - 2e_2 e_3^2 + e_3^2 = 0$$

$$\Rightarrow e_1 + c_2 e_2 + c_3 e_3 - e_2(c_1 + c_2 e_2 + c_3 e_3) = (c_2 - c_1)(c_2 - 1) = 0$$

$$e_2 \neq 1 \Rightarrow c_1 = c_2 \quad (\in k, \text{ scalars}) \Rightarrow e_3^2 = c(1 + e_2) + c_3 e_3$$

$$A \simeq k[x, y] / (x^2 - 1, xy - y, y^2 - c(1+x) - c_3 y)$$

$$3) \quad e_2^2 = 0 \Rightarrow b_1 e_2 + b_3 (b_1 + b_2 e_2 + b_3 e_3) = b_1 b_3 + (b_1 + b_2 b_3) e_2 + b_3^2 e_3 = 0$$

$$\text{Again } b_3^2 = 0, \text{ i.e. } b_3 = 0 \Rightarrow b_1 = 0, \text{ i.e. } e_2 e_3 = b_2 e_2$$

$$\text{Scaling } e_3 \text{ gives } e_2 e_3 = e_2 \quad \text{or} \quad e_2 e_3 = 0 \Leftrightarrow e_2(c_3 - 1) = 0$$

$$\text{If } e_3^2 = c_1 + c_2 e_2 + c_3 e_3, \text{ then } e_3^2 e_2 = c_1 e_2 + c_3 e_2 e_3 = c_1 e_2 = 0$$

$$\text{Thus } e_3^2 = c_2 e_2 + c_3 e_3$$

$$A \simeq k[x, y] / (x^2, xy, y^2 - c_2 x - c_3 y)$$