Def

R ring, M R-module Noetherian if every submodule is fin. gen.

$\Updownarrow$

ACC $M_0 \subset M_1 \subset M_2 \subset \cdots$ stabilizes

$\Updownarrow$

max c Every non-empty set of submodules contains a maximal one under $\subseteq$.

(Usefull lemma.

R ring, M R-module $\supset M_2 \supseteq M_1$. If $\exists N \subset M$ such that

1) $M_1 \cap N = M_2 \cap N$

2) $M_1 + N/N = M_2 + N/N$

Then $M_1 = M_2$

Doesn't need

Pf Let $m_2 \in M_2$. Then $\overset{\text{by 2)}}{\text{modulo}}$ N $\exists\, m_1 \in M_1$ and $n \in N$ s.t. $m_2 \equiv m_1 + n \mod N$, i.e. $n = m_2 - m_1 \in N$. But $M_1 \subset M_2$, thus $m_1 \in M_2$ and $n = m_2 - m_1 \in M_2$, i.e.

$n = m_2 - m_1 \in M_2 \cap N = M_1 \cap N \Rightarrow m_2 = n + m_1 \in M_1$ $\square$

Alt

$$\begin{array}{ccccccccc}
 & & & & m & \longmapsto & m+0 & & \\
0 & \to & M_1 \cap N & \longrightarrow & M_1 & \longrightarrow & M_1+N/N & \longrightarrow & 0 \\
 & & \| & & \cap & & \| & & \\
0 & \to & M_2 \cap N & \longrightarrow & M_2 & \longrightarrow & M_2+N/N & \longrightarrow & 0
\end{array}$$

must be equal)

Proposition.

R ring, M R-module $\supset N$

1) M fin. gen $\iff$ N, M/N fin. gen

2) M Noetheric $\iff$ N, M/N Noetherian

Pf. M Noeth. $\Rightarrow$ Every submod. of N is f.g. $\varphi : M \to M/N \supset P$

$\varphi^{-1}(P)$ fin. gen. $\Rightarrow P$ fin. gen

N, M/N Noeth., $P \subset M$. $P \cap N$, $P+N/N$ fin. gen.. by assumption

$$0 \to P \cap N \to P \to P+N/N \to 0$$

1) $\Rightarrow$ f.g.

$$\underset{\text{Noeth}}{M_1, \dots, M_n} \iff \underset{\text{Noetherian.}}{M_1 \oplus \dots \oplus M_n}$$

Pf. For $n=2$.

$$\begin{array}{ccccccccc} 0 \to & M_1 & \to & M_1 \oplus M_2 & \to & M_2 & \to 0 \\ & \shortparallel & & \shortparallel & & \shortparallel & \\ & M_1 & & M & & M/M_1 & \end{array}$$

Follows from Prop.

Artinian algebras:

$(A, \underline{m})$ local Artinian f.g. k-algebra.

By Weak Nullstellensatz or Zariski's Lemma

$$k \longrightarrow A \longrightarrow A/\underline{m}$$

field
finite alg. ext. of k

$\underset{\text{local Artinia}}{A} \Rightarrow \underline{m}^n = 0$ for some n

$\Rightarrow A \supset \underline{m} \supset \underline{m}^2 \supset \dots \supset \underline{m}^n = 0$

$\underset{\text{Noeth.}}{A} \Rightarrow \underline{m}^i/\underline{m}^{i+1}$ f.g. $A/\underline{m}$-module

$\Rightarrow A$ finite dimensional$/k$.

$\dim_k A = 1$. $\quad A \simeq k$.

$\dim_k A = 2 \quad A \simeq k[X]/(X^2+aX+b) \quad \mathcal{D} = a^2 - 4b$.

$\mathcal{D} = 0$: one root : $k[X]/(X^2)$

$\mathcal{D} \neq 0$: two roots : $k[X]/(X^2-1) \simeq k \oplus k \quad 1 \mapsto (1,1),\ X \mapsto (1,-1)$

$\dim_k A = 3$. $\quad A = \langle 1, e_2, e_3 \rangle$ as k-vector space

$e_2^2 = a_1 + a_2 e_2 + a_3 e_3$. By coordinate change

1) $e_2^2 = e_3$ 2) $e_2^2 = 1$ 3) $e_2^2 = 0$

$e_2 e_3 = b_1 + b_2 e_2 + b_3 e_3$

1) $e_2^3 - b_3 e_2^2 - b_2 e_2 - b_1 = 0 \quad k[X]/(X^3 - b_3X^2 - b_2X - b_1)$

2) $e_2^2 = 1 \Rightarrow (b_3^2 - 1) e_3 + (b_1 + b_2 b_3) e_2 + (b_2 + b_1 b_3) = 0$

If $b_3^2 \neq 1$, then $e_3 \in \langle 1, e_2 \rangle$ and $\dim_k A = 2$. Can assume $b_3 = 1$

$\Rightarrow \quad (b_1+b_2)e_2 + (b_1+b_2) 1 = 0 \quad \Rightarrow \quad b_1 = -b_2$

$\Rightarrow \quad e_2 e_3 = b - b e_2 + e_3 \quad$ or $\quad (e_3 + b) e_2 = e_3 + b$

By coord change $\quad e_2 e_3 = e_3$

Put $e_3^2 = c_1 + c_2 e_2 + c_3 e_3$ and $(e_2 e_3 - e_3)^2 = e_3^2 - 2 e_2 e_3^2 + e_3^2 = 0$

$\Rightarrow \quad c_1 + c_2 e_2 + c_3 e_3 - e_2(c_1 + c_2 e_2 + c_3 e_3) = (c_2 - c_1)(e_2 - 1) = 0$

$e_2 \neq 1 \Rightarrow c_1 = c_2$ ($\in k$, scalars) $\Rightarrow e_3^2 = c(1+e_2) + c_3 e_3$

$$A \simeq k[x,y] \big/ (x^2-1,\ xy-y,\ y^2 - c(1+x) - c_3 y)$$

3) $e_2^2 = 0 \Rightarrow \quad b_1 e_2 + b_3(b_1 + b_2 e_2 + b_3 e_3) = b_1 b_3 + (b_1 + b_2 b_3) e_2 + b_3^2 e_3 = 0$

Again $b_3^2 = 0$, i.e. $b_3 = 0 \Rightarrow b_1 = 0$, i.e. $e_2 e_3 = b_2 e_2$

Scaling e_3 gives $e_2 e_3 = e_2$ or $e_2 e_3 = 0 \Leftrightarrow e_2(e_3 - 1) = 0$

If $e_3^2 = c_1 + c_2 e_2 + c_3 e_3$, then $e_3^2 e_2 = c_1 e_2 + c_3 e_2 e_3 = c_1 e_2 = 0$

Thus $e_3^2 = c_2 e_2 + c_3 e_3$

$$A \simeq k[x,y] \big/ (x^2,\ xy,\ y^2 - c_2 x - c_3 y)$$