Motivation:

$(n) \subset \mathbb{Z}$, $n = p_1^{r_1} \cdots p_m^{r_m}$

Then $(n) = (p_1)^{r_1} \cap (p_2)^{r_2} \cap \cdots \cap (p_m)^{r_m}$, $\sqrt{(p_i)^{r_i}} = (p_i)$

Can we do the same in general?

Ex. $I = (x^2, xy) \subset k[x,y] \Rightarrow I = (x) \cap (x^2, y+\alpha x)$ $\alpha \in k$

1) $x^2 \in (x) \cap (x^2, y+\alpha x)$

2) $xy \in (x) \cap (x^2, y+\alpha x) \Leftarrow xy = x(y+\alpha x) - \alpha x^2$

3) $f x^2 + g(y+\alpha x) \in (x)$

$\Rightarrow g \in (x)$; $g = h \cdot x$

$\Rightarrow f x^2 + g(y+\alpha x) = f x^2 + hxy + \alpha h x^2 \in (x^2, xy)$

$\alpha \neq \beta \Rightarrow (x^2, y+\alpha x) \neq (x^2, y+\beta x)$

Suppose $y + \alpha x \in (x^2, y+\beta x)$

$\Rightarrow (y+\alpha x) - (y+\beta x) = (\alpha - \beta) x \in (x^2, y+\beta x)$

$\Rightarrow x \in (x^2, y+\beta x) \Rightarrow y \in (x^2, y+\beta x)$ Contradiction $y \notin (x) \cap (x^2, y+\beta x)$

$\sqrt{(x^2, y+\alpha x)} = (x, y) \Rightarrow \sqrt{I} = (x, y)$

Associated primes:

$\exists\, m \in k[x,y]/I$ s.t. $p = \mathrm{Ann}(m)$

Put $m = x$, then $\mathrm{Ann}(x) = (x, y)$

Put $m = x+y$, then $\mathrm{Ann}(x) = (x)$.

Associated Primes

R ring, M module. $\quad \mathfrak{p} \in \mathrm{Ass}(M) \overset{\text{def}}{\iff} \exists\, m \in M$ s.t. $\mathfrak{p} = \mathrm{Ann}(m)$

Notice 1) $\mathfrak{p} \in \mathrm{Ass}(M) \iff \exists\ \underset{\text{R-lin. inj.}}{R/\mathfrak{p} \hookrightarrow M}$

Pf. If $\mathfrak{p} = \mathrm{Ann}(m)$, then $\begin{array}{c} R/\mathfrak{p} \longrightarrow M \\ 1 \longmapsto m \end{array}$ is injective

If $R/\mathfrak{p} \overset{\varphi}{\hookrightarrow} M$, let $m = \varphi(1)$. Then $\mathrm{Ann}(m) = \mathfrak{p}$.

2) $\mathrm{Ass}(R/\mathfrak{p}) = \{\mathfrak{p}\}$

Pf. Let $x \in R$, $x \notin \mathfrak{p}$. Then $\mathrm{Ann}(x) = \mathfrak{p}$ (prime).

Ex. 1) $R = \mathbb{Z}$ $\quad M = \mathbb{Z}/(N)$ $\quad \mathrm{Ass}(M) = \{(p) \mid p \mid N\}$

2) $R = k[x,y]$ $\quad M = k[x,y]/(x^2, xy)$

$f \in M \quad \mathrm{Ann}(f) = ?$

$f \cdot g = F x^2 + G \cdot xy = x(Fx + Gy)$

If $x \mid f \Rightarrow \quad g(a_0 + x \cdot f_2 + y \cdot g_2) \in (x,y)$

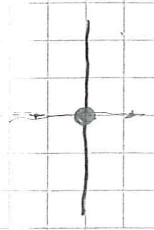
$f = x \cdot f_1 = x(a_0 + x f_2 + y \cdot g_2)$

If $a_0 = 0$, then ok $\Rightarrow f = x \cdot f_1 \in (x^2, xy)$

If $a_0 \neq 0$, then $g \in (x,y)$

If $x \nmid f \Rightarrow x \mid g$ i.e. $g \in (x)$

$\Longrightarrow \mathrm{Ass}(M) = \{(x), (x,y)\}$



$(x^2, xy) = (x) \cap (x^2, y + \alpha x)$

Zero-set: $x = 0$, but $x = y = 0$ is a double point

$(x^2 - t, xy)$: Zero set: $(\sqrt{t}, 0), (-\sqrt{t}, 0)$ 2 points

But (,) —"—

Def $\underset{\text{ring}}{R} \ni x$ $\quad x$ is nilpotent on M if $\exists\, n \geq 1$ s.t.

$$x^n \cdot m = 0 \quad \forall\, m \in M$$

Set of nilpotents on M: $\mathrm{nil}(M) = \sqrt{\mathrm{Ann}(M)}$

Properties: 1) $\underset{f.g/R}{M} \Rightarrow \mathrm{nil}(M) = \bigcap_{\mathfrak{p} \in \mathrm{Supp}(M)} \mathfrak{p}$

Pf In general $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a} \subset \mathfrak{p}} \mathfrak{p}$ (Schwnullstellensatz)

$$\Rightarrow \sqrt{\mathrm{Ann}(M)} = \bigcap_{\mathrm{Ann}(M) \subset \mathfrak{p}} \mathfrak{p}$$

But $\mathfrak{p} \in \mathrm{Supp}(M) \Leftrightarrow M_{\mathfrak{p}} \neq 0 \Leftrightarrow \mathrm{Ann}(M) \cap R \setminus \mathfrak{p} \neq \emptyset$

$\Leftrightarrow \mathrm{Ann}(M) \subset \mathfrak{p}$.

R Noetherian 2) $\underset{f.g/R}{M} \Rightarrow \mathrm{nil}(M) = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$

Pf. $\underset{\text{Noeth.}}{R} \Rightarrow \mathrm{Ass}(M) \subset \mathrm{Supp}(M)$

Pf. Let $\mathfrak{p} = \mathrm{Ann}(m) \in \mathrm{Ass}(M)$. Assume $M_{\mathfrak{p}} = 0$

Then $\exists\, x \in R - \mathfrak{p}$ s.t. $xm = 0 \Rightarrow x \in \mathrm{Ann}(m)$

$\Rightarrow x \in \mathfrak{p}$ and $x \notin \mathfrak{p}$.

If $\mathfrak{q} \in \mathrm{Supp}(M)$, then $\exists\, \mathfrak{p} \in \mathrm{Ass}(M)$; $\mathfrak{p} \subset \mathfrak{q}$.

Pf $M_{\mathfrak{q}} \neq 0 \Leftrightarrow \mathrm{Ass}_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}) \neq \emptyset$

$S = R \setminus \mathfrak{q}$, $\Rightarrow \mathrm{Ann}\left(\frac{m}{t}\right) = S^{-1}\mathfrak{p}$, $\mathfrak{p} = (x_1, .., x_n)$

$\Rightarrow \frac{x_i m}{t} = 0$, i.e. $\exists\, s_i \in S$ s.t. $s_i x_i m = 0$

Let $s = \prod s_i \Rightarrow x_i \in \mathrm{Ann}(sm) \Rightarrow \mathfrak{p} \subseteq \mathrm{Ann}(sm)$

Let $b \in \mathrm{Ann}(sm)$. Then $\frac{bsm}{st} = 0 \Rightarrow \frac{b}{1} \in S^{-1}\mathfrak{p}$

and $b \in \mathfrak{p} \Rightarrow \mathfrak{p} = \mathrm{Ann}(sm) \Rightarrow \mathfrak{p} \in \mathrm{Ass}(M)$

since $S^{-1}\mathfrak{p}$ prime

and $\mathfrak{p} \cap (R - \mathfrak{q}) = \emptyset$.

$\Rightarrow \mathfrak{p} \subset \mathfrak{q}$

Def

R ring, M R-module $\supset Q$. If $\mathrm{Ass}(M/Q) = \{\mathfrak{p}\}$, then Q is $\mathfrak{p}$-primary in M

Ex $R=\mathbb{Z}$, $M=\mathbb{Z}$, $Q=(p^m)$ $\Rightarrow$ $\mathrm{Ass}(\mathbb{Z}/(p^m)) = \{(p)\}$

In fact $\mathrm{Ann}(N) = \{x\in\mathbb{Z} \mid xN \in (p^m)\} \overset{!}{=} (p^r)$

where $p^{m-r} \mid N$ and is the highest power of p

Thus $N = p^{m-1} \Rightarrow \mathrm{Ann}(N) = (p)$ (prime)

Equivalent definition: $Q\subset M$, $\mathfrak{p} = \bigcap_{\mathfrak{q}\in \mathrm{Ass}(M/Q)} \mathfrak{q} = \mathrm{nil}(M/Q)$

Let $r\in R$, $m\in M$ s.t. $rm\in Q$, but $m\notin Q$. Then $r\in\mathfrak{p}$

Pf. $r\cdot m\in Q \Rightarrow r$ is a zero-divisor in M/Q.

Thus $\mathrm{Ann}(x) \subset \mathfrak{p}$ ($\because \mathrm{Ann}(x)$ consists of zero-divisors)

~~But~~ $\mathfrak{p}^M = \bigcap_{\mathfrak{q}\in\mathrm{Ass}(M/Q)} \mathfrak{q} \subset \bigcap \mathrm{Ann}(x) \subset \mathfrak{p}$

$\Rightarrow$ Equality $\Rightarrow r\in\mathfrak{p}$

Goal:

R ring, M R-module $\supset N$. Primary decomposition:

$N = Q_1 \cap \cdots \cap Q_r$ $\quad Q_r$ primary

Minimal

1) $N \neq \bigcap_{i\neq j} Q_i \quad \forall j$

2) Q_i : $\mathfrak{p}_i$-primary $\Rightarrow \mathfrak{p}_1,\dots,\mathfrak{p}_r$ distinct.

For a Noetherian module M, $\mathfrak{p}\subset R$ prime, Q_1, Q_2 $\mathfrak{p}$-primary

$\Rightarrow Q_1\cap Q_2$ $\mathfrak{p}$-primary

WC chess 1894-1921

Existence:

Lasker – Noether.

1905 polynomial rings

1921 General case

Over a Noetherian ring, each proper submodule of a fin.gen. submodule has a minimal primary decomposition

Uniqueness 1. Result:

R ring, M R-module

$N = Q_1 \cap \ldots \cap Q_r$ minimal primary decomp.

$\sqrt{Q_i} = \mathfrak{p}_i$

$\Rightarrow$ $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ uniquely determined

in fact $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\} = \mathrm{Ass}(M/N)$

2. Result

R ring Noeth., $M \supset N$ R-module fin. gen.

$\mathfrak{p} \in \mathrm{Ass}(M/N)$ Minimal

$\Rightarrow$ The $\mathfrak{p}$-primary component Q in a minimal dec. is uniquely determined,

in fact $Q = N^S$ where $S = R - \mathfrak{p}$

$$N^S = \{m \in M \mid \exists s \in S \text{ with } sm \in N\}$$

$$M \to M/N \to S^{-1}(M/N) \qquad = \ker \beta$$

$\beta(m) = \frac{\bar{m}}{1} = 0$ if $\exists s \in S$ s.t. $ms \in N$