Lemma 1.

R ring, M R-module, $N = Q_1 \cap \dots \cap Q_r$ primary decomp., $\mathrm{Ass}(M/Q_i) = \{\mathfrak{p}_i\}$ $\Rightarrow$ $\mathrm{Ass}(M/N) \subseteq \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$

If equality and $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ distinct, then minimal decomp.

Converse holds if R is Noetherian

Pf. $\varphi : M/N \longrightarrow \bigoplus M/Q_i$ $\Rightarrow$ $\ker \varphi = \bigcap Q_i = N$

$\Rightarrow$ φ injective

$\Rightarrow$ $\mathrm{Ass}(M/N) \subseteq \bigcup \mathrm{Ass}(M/Q_i) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$.

If $N = Q_1 \cap \dots \cap Q_r = Q_2 \cap \dots \cap Q_r$, then

$\mathrm{Ass}(M/N) \subseteq \{\mathfrak{p}_2, \dots, \mathfrak{p}_r\}$

Equality + $\mathfrak{p}_i$ distinct $\Rightarrow$ $N = Q_1 \cap \dots \cap Q_r$ minimal

If $N = \bigcap Q_i$ is minimal, put $P_j = \bigcap_{i \neq j} Q_i$

$\Rightarrow$ $P_j \cap Q_j = N$ (1)

$P_j / N \neq 0$ (2)

$$\begin{tikzcd} P_j/N \arrow[r, hook, "\iota_2"] \arrow[d, hook, "\iota_1"', "\overset{(1)}{\cap P_j}" near start] & M/N \\ M/Q_j & \end{tikzcd}$$

R Noeth $\overset{(2)}{\Rightarrow}$ $\mathrm{Ass}(P_j/N) \neq \emptyset$. ι_1 is injective and $\mathrm{Ass}(M/Q_j) = \{\mathfrak{p}_j\}$

$\Rightarrow$ $\mathfrak{p}_j \in \mathrm{Ass}(P_j/N)$

ι_2 injective $\Rightarrow$ $\mathfrak{p}_j \in \mathrm{Ass}(M/N)$ $\forall j$

$\Rightarrow$ $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\} \subset \mathrm{Ass}(M/N)$ $\square$

Consequence: 1st uniquenes theorem

Lemma 2.

M R-module, $\mathfrak{p} \in \mathrm{Ass}(M)$ $\Rightarrow$ $\exists N \subset M$ s.t. 1) $\mathrm{Ass}(M/N) = \{\mathfrak{p}\}$

2) $\mathrm{Ass}(N) = \mathrm{Ass}\{M\} \setminus \{\mathfrak{p}\}$

Pf A) Consider sequence of nested submodules

$N_1 \subset N_2 \subset \dots \subset M$

and let $N = \bigcup N_i$.

Let $\mathfrak{q} \in \mathrm{Ass}(N)$, say $\mathfrak{q} = \mathrm{Ann}(x)$, $x \in N$.

$N = \bigcup N_i \Rightarrow x \in N_i$ for some i

$\Rightarrow \mathfrak{q} \in \mathrm{Ass}(N_i)$ for some i

$\Rightarrow \mathrm{Ass}(N) \subset \bigcup_i \mathrm{Ass}(N_i)$

But $N_i \subset N \Rightarrow \mathrm{Ass}(N_i) \subset \mathrm{Ass}(N)$

$\Rightarrow \mathrm{Ass}(N) = \bigcup_i \mathrm{Ass}(N_i)$

B) Let $S = \{N \in M \mid \mathrm{Ass}(N) \subset \mathrm{Ass}(M) \setminus \{\mathfrak{p}\}\} \neq \emptyset$ $(0 \in S)$

Every sequence of nested submodules is bounded in S by A).

$\Rightarrow \exists$ max element N s.t. $\mathrm{Ass}(N) \subset \mathrm{Ass}(M) \setminus \{\mathfrak{p}\}$

C) $0 \to N \to M \to M/N \to 0 \Rightarrow \mathrm{Ass}(M) \subset \mathrm{Ass}(N) \cup \mathrm{Ass}(M/N)$

$\mathfrak{p} \in \mathrm{Ass}(M)$, but $\mathfrak{p} \notin \mathrm{Ass}(N) \Rightarrow \mathfrak{p} \in \mathrm{Ass}(M/N)$

D) If $\mathfrak{q} \in \mathrm{Ass}(M/N)$, then $\exists N \subset N' \subset M$ s.t. $N'/N \simeq R/\mathfrak{q}$

$0 \to N \to N' \to N'/N \simeq R/\mathfrak{q} \to 0 \Rightarrow \mathrm{Ass}(N') \subset \mathrm{Ass}(N) \cup \{\mathfrak{q}\}$

Since N is max s.t. $\mathrm{Ass}(N) \subset \mathrm{Ass}(M) \setminus \{\mathfrak{p}\}$

and $N \subsetneq N'$, $\mathfrak{p} \in \mathrm{Ass}(N')$

But $\mathfrak{p} \notin \mathrm{Ass}(N) \Rightarrow \mathfrak{p} = \mathfrak{q}$.

Pf Lasker - Noether theorem $N \subsetneq M$ $\exists$ $N = \underset{\text{minimal}}{Q_1 \cap \ldots \cap Q_r}$

Let $\mathrm{Ass}(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$

By lemma 2. $\exists$ $\mathfrak{p}_i$-primary $Q_i \subset M$ s.t. $\mathrm{Ass}(Q_i/N) = \mathrm{Ass}(M/N) \setminus \{\mathfrak{p}_i\}$

Let $P = \bigcap Q_i$

For each i we have $P/N \subset Q_i/N \Rightarrow \mathrm{Ass}(P/N) \subset \mathrm{Ass}(Q_i/N)$

$\Rightarrow \mathrm{Ass}(P/N) \subset \bigcap_i \mathrm{Ass}(Q_i/N) = \bigcap_i (\mathrm{Ass}(M/N) \setminus \{\mathfrak{p}_i\}) = \emptyset$

$\Rightarrow P/N = 0 \Rightarrow P = N$

By Lemma 1 decomposition is minimal

Lemma 3

$S \subset R$ M

mc Noeth. f.g module/R

$N = Q_1 \cap \ldots \cap Q_r \subset M$
minimal decomposition of N

Q_i is $\mathfrak{p}_i$-primary

$S \cap \mathfrak{p}_i = \emptyset$ for $i = 1, 2, \ldots, h < r$.

$\Rightarrow$ 1) $S^{-1}N = S^{-1}Q_1 \cap \ldots \cap S^{-1}Q_h \subset S^{-1}M$

2) $N^S = \{m \in M \mid \exists s \in S \text{ s.t. } sm \in N\}$
$= Q_1 \cap \ldots \cap Q_h$.

Pf 1) Localization commutes with intersection

$\Rightarrow S^{-1}N = S^{-1}Q_1 \cap \ldots \cap S^{-1}Q_r = S^{-1}Q_1 \cap \ldots \cap S^{-1}Q_h$

since $S^{-1}Q_i = S^{-1}M$ when $S \cap \mathfrak{p}_i \neq \emptyset$.

and $S^{-1}Q_i$ is $S^{-1}\mathfrak{p}_i$-primary. Hence 1)

$\mathfrak{p}_i$ distinct $\Rightarrow$ $S^{-1}\mathfrak{p}_i$ distinct

2) We have $\varphi_S^{-1}(S^{-1}P) = P^S$, thus

$N^S = \varphi_S^{-1}(S^{-1}Q_1 \cap \ldots \cap S^{-1}Q_h) = Q_1^S \cap \ldots \cap Q_h^S$

But $Q_i^S = Q_i$. Hence 2)

$\Updownarrow$

$Q_i \subset Q_i^S$, $m \in Q_i^S \Rightarrow \exists s \in S$ s.t. $sm \in Q_i$

Obvious since $1 \in S$ $s \notin \mathfrak{p} \Rightarrow m \in Q_i$

2^{nd} uniqueness theorem follows by $S = R - \mathfrak{p}_i$.

Ex. of consequence: Krull intersection

$\mathfrak{a} \subset R$ Noeth. $\quad M$ f.g./$R \quad N = \bigcap_{n \geq 0} \mathfrak{a}^n M \Rightarrow \exists x \in \mathfrak{a}$ s.t. $(1+x)N = 0$

Pf. By Cayley-Hamilton $\exists x \in \mathfrak{a}$ iff. $\mathfrak{a}N = N$

1) $\mathfrak{a}N \subset N$ is obvious

2) Let $\mathfrak{a}N = Q_1 \cap \dots \cap Q_r$ (primary decomp)
Q_i $\mathfrak{p}_i$-primary

If $(a \in \mathfrak{a} \setminus \mathfrak{p}_i \neq \emptyset)$ then $a \cdot N \subset Q_i$ (by def of primary)

and if $an \in Q_i$, $a \notin \mathfrak{p}_i$, then $n \in Q_i$

$\Rightarrow N \subset Q_i$

If $\mathfrak{a} \subset \mathfrak{p}_i$, then since $(\sqrt{\mathfrak{a}})^{n_i} \subset \mathfrak{a}$ for big n_i
$\mathfrak{a}$ f.g.

$\Rightarrow \mathfrak{a}^{n_i} M \subset Q_i$

$\Rightarrow N \subset Q_i$

$\Rightarrow N \subset \bigcap Q_i = \mathfrak{a}N$ and $N = \mathfrak{a}N$. $\square$