1 a) $\varphi : \mathbb{Z}[X]/(X^2+5) \longrightarrow \mathbb{Z}[i\sqrt{5}]$
$X \longmapsto i\sqrt{5}$

$P(X) \longmapsto P(i\sqrt{5}) = 0$
$\Rightarrow X^2+5 \mid P(X)$

b) $2 \cdot 3 = (1+i\sqrt{5})(1-i\sqrt{5})$

Irreducible $\|a+i\sqrt{5}\,b\|^2 = a^2+5b^2 \geq 1$ and $\|a+i\sqrt{5}\,b\|^2 \in \mathbb{Z}$

$\|2\|^2 = 4$, $\|3\|^2 = 9$, $\|1+i\sqrt{5}\|^2 = 6$ $\|1-i\sqrt{5}\|^2 = 6$

If any of the four are products, then product of square norms must be less than the given norms. Impossible

c) $\mathfrak{a} = (2, 1+i\sqrt{5}) \subset \mathbb{Z}[i\sqrt{5}]$,

$\Rightarrow \psi : \mathbb{Z}[i\sqrt{5}]/\mathfrak{a} \longrightarrow \mathbb{Z}/(2)$ iso

$i\sqrt{5} \longmapsto 1$
$1+i\sqrt{5} \longmapsto 0$
$2 \longmapsto 0$

$\ker \psi \equiv a+i\sqrt{5}\cdot b \mapsto a+b \equiv 0 \ (2)$, $a,b \in \mathbb{Z}$

$\Rightarrow a \in (2)$ or $b \in (2)$ or $a \equiv b \equiv 1 \ (2)$
$\Downarrow$
$a+i\sqrt{5}\,b = 1+i\sqrt{5}+(2)$

$\mathfrak{a}^2 = (2, 1+i\sqrt{5})^2 = (4, 2+2i\sqrt{5}, 1+2i\sqrt{5}-5)$
$= (4, 2+2i\sqrt{5}, -4+2i\sqrt{5}) = (2, 2i\sqrt{5}) = (2)$

$r(\mathfrak{a}^2) =$ Notice $(a+i\sqrt{5}\,b)^2 = a^2-5b^2+2i\sqrt{5}\,b \in (2)$

$\Leftrightarrow a+b \equiv 0 \ (2)$

i.e. $r(2) = (2, 1+i\sqrt{5}) = \mathfrak{a}$

d) $\mathrm{Frac}(\mathbb{Z}[i\sqrt{5}]) \subset \mathbb{Q}(i\sqrt{5})$ since $\dfrac{a+i\sqrt{5}\,b}{c+i\sqrt{5}\,b} \in \mathbb{Q}(i\sqrt{5})$

But $\mathbb{Q} \subset \mathrm{Frac}(\mathbb{Z}[i\sqrt{5}])$ and

$\frac{r}{s} i\sqrt{5} = \dfrac{r\,i\sqrt{5}\cdot i\sqrt{5}}{s \cdot i\sqrt{5}} = \dfrac{-5r}{s \cdot i\sqrt{5}} \in \mathrm{Frac}(\mathbb{Z}[i\sqrt{5}])$

$\Rightarrow \mathrm{Frac}(\mathbb{Z}[i\sqrt{5}]) = \mathbb{Q}(i\sqrt{5})$

Let $x = a+b\sqrt{-5} \in \mathbb{Q}(\sqrt{-5})$ be integral $/ \mathbb{Z}[\sqrt{-5}]$; $g(x) = 0$.

x satisfies monic equation $f(x) = x^2 - 2ax + (a^2+5b^2) \in \mathbb{Q}[x]$

We know that $a-b\sqrt{-5}$ integral $/\mathbb{Z}\sqrt{-5} \Rightarrow 2a$ integral $/\mathbb{Q}$

$\Rightarrow a \in \frac{1}{2}\mathbb{Z}$. Also $2b\sqrt{-5}$ integral $\Rightarrow -4b^2 \cdot 5$ integral$/\mathbb{Q}$

$\Rightarrow b^2 \in \frac{1}{20}\mathbb{Z} \Rightarrow b \in \frac{1}{2}\mathbb{Z} \Rightarrow x = \frac{a}{2} + \frac{b}{2}\sqrt{-5}$, $a, b \in \mathbb{Z}$

But $(\frac{a}{2})^2 + 5(\frac{b}{2})^2 = \frac{a^2+5b^2}{4} = \frac{a^2}{4} + \frac{b^2}{4} + b^2 \in \mathbb{Z} \Leftrightarrow a^2 + b^2 \equiv 0 \ (4)$

Only for $a, b \equiv 0 \ (2)$

2. a) $(0) \neq \underline{m} \in A \subset K$ local field

We have $\underline{m}K = K$ $(0 \neq x \in \underline{m} \Rightarrow \frac{1}{x} \in K)$

K fin. gen. as A-module

Nakayama $\underline{m}K = K \Rightarrow K = 0$. But $A \neq 0$.

b) $k \subset R = k[x,y]$

$I = (x,y)^n$

$f : I \to I$ surjective

By Cor 10.4. and I is fin gen R-module

$\Rightarrow f$ isomorphism

c) $S \subset R$

M fin.gen. $/R$.

$S^{-1}M = 0 \Leftrightarrow \forall m_i \in M, \exists s_i \in S$ s.t. $sm = 0$

$M = \langle m_1, .., m_r \rangle$ Let $s = \prod s_i \Rightarrow sm = \prod s_i m = 0$.

d) $\mathfrak{a} \subset R$

$S = 1 + \mathfrak{a}$.

$(1+x)(1+y) = 1 + x + y + xy \in 1 + \mathfrak{a}$. M.C

Suppose $\exists \frac{a}{s} \in S^{-1}\mathfrak{a}$ $\frac{a}{s} \notin rad(S^{-1}R)$

$\Leftrightarrow \exists \frac{r}{t} \in S^{-1}R$ s.a. $1 - \frac{a}{s}\frac{r}{t}$ is a non-unit in $S^{-1}R$

We have $1 - \frac{a}{s}\frac{r}{t} = \frac{st - ar}{st}$. Now $s, t \in S = 1 + \mathfrak{a}$

$\Rightarrow st - ar \in (1+\mathfrak{a})(1+\mathfrak{a}) + \mathfrak{a} = 1 + \mathfrak{a} = S$

$\Rightarrow 1 - \frac{a}{s}\frac{r}{t}$ is a unit in $S^{-1}R$ Contradiction

e) M fin gen$/R$ $\mathfrak{a} \subset R$ Put $S = 1 + \mathfrak{a}$. Then $S^{-1}\mathfrak{a} \subset rad(S^{-1}R)$.

$\mathfrak{a}M = M \Rightarrow S^{-1}\mathfrak{a} \cdot S^{-1}M = S^{-1}M$. Nakayama $\Rightarrow S^{-1}M = 0$

$\Updownarrow$

$\exists s \in S$ s.t. $sM = 0$

$(s = 1 + x, x \in \mathfrak{a}.)$

3. $n \in \mathbb{Z}$ $S = \{m \in \mathbb{Z} \mid (m,n) = 1\}$.

$\mathbb{Z}_{(n)} := S^{-1}\mathbb{Z}$

a) $n = 6$. $\mathbb{Z}_{(6)}$: Every prime except 2,3 are invertible

$\Rightarrow \underline{m}_1 = (2)\mathbb{Z}_{(6)}$, $\underline{m}_2 = (3)\mathbb{Z}_{(6)}$

b) $\mathbb{Z}_{(6)}/\underline{m} = S^{-1}\mathbb{Z}/_{S^{-1}(2)} = S^{-1}\mathbb{Z}/_{(2)} = \mathbb{Z}/_{(2)}$

c) $\mathcal{R}\text{-rad}(\mathbb{Z}_{(6)}) = (2)\mathbb{Z}_{(6)} \cap (3)\mathbb{Z}_{(6)} = S^{-1}((2)\cap(3)) = S^{-1}(6)$

$$\mathbb{Z}_{(6)}/\mathcal{R} = S^{-1}\mathbb{Z}/S^{-1}(6) = S^{-1}\mathbb{Z}/(6)$$

$S = \{1,-1\}$ already units $= \mathbb{Z}/(6)$

d) $\mathbb{Z}_{(n)}$, $n = p_1^{r_1}\cdots p_s^{r_s} \Rightarrow (p_i)\mathbb{Z}_{(n)}$ $i = 1,\dots,s$
max ideals

$\Rightarrow \mathbb{Z}_{(n)}/(p_i)\mathbb{Z}_{(n)} = \underline{\mathbb{Z}/(p_i)}$

4.

a) $\mathbb{Z}/(m) \otimes \mathbb{Z}/(n) \longrightarrow \mathbb{Z}/(d)$ $d = \gcd(m,n)$

$a\otimes b \longmapsto a\cdot b$ &

$1\otimes 1 \longmapsto 1$ surjective.

$a\otimes b \longmapsto 0$ i.e. $ab \equiv 0(d)$

$\shortparallel$

$1\otimes ab$ or $ab = spn + sqm$.

$\shortparallel$

$1\otimes spn + sqm$

$\shortparallel$

$1\otimes sqm = m\otimes sq = 0$.

b) $0 \longrightarrow P \longrightarrow Q$ $\otimes M \otimes_R N$

$\Downarrow$ M flat

$0 \longrightarrow P\otimes_R M \longrightarrow Q\otimes_R M$

$\Downarrow$ N flat

$$0 \longrightarrow (P\otimes_R M)\otimes_R N \longrightarrow (Q\otimes_R M)\otimes_R N$$
$$\shortparallel \qquad\qquad \shortparallel$$
$$P\otimes(M\otimes_R N) \longrightarrow Q\otimes_R(M\otimes_R N)$$

c) $V^* = \mathrm{Hom}_k(V,k)$, $\varphi: V^*\otimes_k V \longrightarrow \mathrm{End}_k(V)$

$V = \langle e_1,\dots,e_n\rangle$ $\varphi(\alpha\otimes v)(w) = \alpha(w)\cdot v$

Dual basis $V^* = \langle e_1^*,\dots,e_n^*\rangle$ where $e_j^*(e_i) = \delta_{ij}$

Let $\beta \in \mathrm{End}_k(V)$, and consider the element

$$\sum_i e_i^* \otimes \beta(e_i) \in V^*\otimes V$$

Then we have

$$\varphi\left(\sum_i e_i^* \otimes \beta(e_i)\right)(w) = \sum_i e_i^*(w)\cdot \beta(e_i)$$
$$= \beta\left(\sum_i e_i^*(w)\cdot e_i\right)$$
$$= \beta(w)$$

and $\sum e_i^* \otimes \varphi(\alpha\otimes v)(e_i) = \sum e_i^* \otimes \alpha(e_i)\cdot v$

$$= \left(\sum \alpha(e_i)\cdot e_i^*\right)\otimes v = \alpha \otimes v$$

(Remember: $\alpha = \sum \alpha_i e_i^*$, $\alpha(e_j) = \sum \alpha_i e_i^*(e_j) = \alpha_j$)

d) Basis for $V \otimes_k W$: $\langle e_i \otimes f_j \rangle$ $i = 1,2,\dots,n$, $j = 1,2,\dots,m$

$V = \langle e_1, \dots, e_n \rangle$ $\dim(V\otimes_k W) = \dim_k V \cdot \dim_k W$

$W = \langle f_1, \dots, f_m \rangle$

5.

a) $x \in R$, $x^n = 0$ for some $n \geq 1$

$1 = x^n + 1 = \underbrace{(x^{n-1} - x^{n-2} + \dots (-1)^n 1)}_{\in R}(x+1)$

b) $k[x,y]/(xy)$ Prime ideals:

Max. $(x-a, y-b)$ $a=0$ or $b=0$

Prime. (x), (y).

c) $\psi : k[x] \xrightarrow{x \mapsto x} k[x,y]/(xy)$ Fiber: $\{\mathfrak{q} \mid \psi^{-1}(\mathfrak{q}) = \mathfrak{p}\}$

$k[x] \ni \mathfrak{p}$, $0 \neq$ prime, $\mathfrak{p} = (x-a)$

$a \neq 0$. $(x-a, y)$

$a = 0$ $(x, y-b)$