Def.

R ring, M module/R

1) M simple : $N \subseteq M \Rightarrow N = 0$ or $N = M$

2) $0 = M_0 \subset \dots \subset M_m = M$ (length $= m$.

composition series if M_i/M_{i-1} simple

3) $\ell(M)$ length $= \inf\{m \mid M$ has composition series of length $m\}$

No composition series $\ell(M) = \infty$, $\ell(0) = 0$.

Jordan-Hölder Theorem

R ring, M module with composition series

1) Every chain of submodules can be refined to a composition series

2) Every composition series is of the same length $\ell(M)$

Pf. A. $M' \subsetneq M \Rightarrow \ell(M') < \ell(M)$ $\quad 0 = M_0 \subset \dots \subset M_m = M$

$$0 = M_0 \subset M_1 \subset \dots \subset M_i \subset M_{i+1} \subset \dots \subset M_m = M$$

$$\cap M' : \quad 0 = M_0' \subseteq M_1' \subseteq \dots \subseteq M_i' \subseteq M_{i+1}' \subseteq \dots \subseteq M_m' = M'$$

$$\frac{M_{i+1}'}{M_i'} \subseteq \frac{M_{i+1}' + M_i}{M_i' + M_i} \subseteq \frac{M_{i+1}}{M_i} \text{ simple}$$

If $M_{i+1}'/M_i' \neq 0 \ \forall i$, then $M' = M$

If not, then $\ell(M') < \ell(M)$

B. $0 = N_0 \subsetneq N_1 \subsetneq \dots \subsetneq N_n = N \subset M \Rightarrow n < \ell(M)$

By induction on $\ell(M)$:

$\ell(M) = 0 \Rightarrow M = 0$ and $n = 0$

Assume $\ell(M) > 0$. If $n = 0$, then ok

If $n \geq 1$, then $\ell(N_{n-1}) < \ell(M) \Rightarrow n-1 < \ell(N_{n-1})$

$\Rightarrow n < \ell(M)$

C. If N_{i+1}/N_i is not simple, we can find
$$N_i \subsetneq N' \subsetneq N_{i+1}$$
$\Rightarrow$ we can refine $\{N_i\}$ into a composition series in finitely many steps.

D. $\underset{\text{composition series}}{\{N_i\}}$ $\Rightarrow$ $\ell(M) \leq n$.

But $n \leq \ell(M)$ $\Rightarrow$ $\ell(M) = n$

$\square$

Additivity of length.

$\underset{\text{ring}}{R}$ $\underset{\text{module}}{M}$ $\qquad 0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$

$\ell(M) < \infty$ $\qquad\qquad \Downarrow$

$$\ell(M) = \ell(M') + \ell(M'')$$

Pf. The chain $0 \subset M' \subset M$ can be refined to a compo. series
$$0 = M_0 \subset M_1 \subset \dots \subset \underset{M_k}{\overset{\|}{M'}} \subset \dots \subset M_m = M$$
$$\Rightarrow 0 = M_k/M' \subset M_{k+1}/M' \subset \quad \subset M/M' = M''$$
$$\Rightarrow \ell(M) = \ell(M') + \ell(M'')$$
(We have $\dfrac{M_{i+1}/M'}{M_i/M'} \cong M_{i+1}/M_i$)

Def. R graded ring if 1) $R = \bigoplus_{n\geq 0} R_n$, R_n additive subgroup

2) $R_m \cdot R_n \subset R_{m+n}$.

$\Rightarrow$ R_0 subring i.e. $1 \in R_0$

Suppose $1 = \sum_{i=0}^{n} x_i$

Then $1 = 1\cdot 1 = \sum_{i,j=0}^{n} x_i \cdot x_j = \sum_{i=0}^{n} x_i$

We have in degree $2n$:

$x_i^2 = 0 \Rightarrow x_i = 0$

Thus $1 = x_0 \in R_0$

Suppose $1 = \sum x_i$, $x_i \in R_i$. Let $z = \sum z_j$; $z_j \in R_j$

Now fix n. Then $z_n = 1\cdot z_n = \sum x_i z_n$

$\Rightarrow \sum_{i>0} x_i z_n = z_n - x_0 z_n \in R_n$

$\Rightarrow x_j z_n = 0 \quad \forall j>0$

$\Rightarrow x_j z = 0 \quad \forall z \in R$ og $j>0$

If $z=1$, then $x_j \cdot 1 = x_j = 0 \quad \forall j>0$

$\Rightarrow 1 = x_0 \in R_0$

Def R graded ring, M graded module if 1) $M = \bigoplus M_i$

2) $R_i \cdot M_j \subset M_{i+j}$.

M graded module $\rightsquigarrow$ $M(n)$ shift module, $(M(n))_i = M_{n+i}$

Lemma

$R = \bigoplus R_n$ fin.gen/R_0 , $M = \bigoplus M_n$ fin.gen/R $\Rightarrow$ M_n fin.gen R_0-module

Homogeneous generators

Pf. $R = R_0[x_1,\dots,x_r]$ $\quad M = \langle m_1,\dots,m_s\rangle$ $\quad m_i \in M_{\ell_i}$

$m \in M_n$: $m = \sum f_i m_i$, $f_i \in R$; $f_i = \sum f_{ij}$, $f_{ij} \in R_j$

Let $f_i' = f_{ik}$, $k = n - \ell_i$ (or 0 if $n < \ell_i$)

$\Rightarrow m = \sum f_i' m_i$ where $f_i' = \sum a_{i_1\dots i_r} x_1^{i_1}\cdots x_r^{i_r} \in R_k$

$\Rightarrow M_n$ is gen/R_0 by $x_1^{i_1}\cdots x_r^{i_r} m_i$

Hilbert functions:

$R = \bigoplus R_n$ $\quad M = \bigoplus M_n$

fin.gen/R_0 $\quad$ graded/R, fin.gen.

R_0 Artinian ($R_0 = k$)

$\ell(M_n) < \infty$

$n \longmapsto \ell(M_n)$

Hilbert function

$$H(M,t) := \sum_{n\in\mathbb{Z}} \ell(M_n)\cdot t^n$$

Hilbert series.

Ex. $R = k[x_1, \dots, x_r]$ $\quad \dim R_n = \binom{r-1+n}{r-1}$

pol. ring.

$$H(R,t) = \frac{1}{(1-t)^r}$$

$$\frac{1}{(1-t)^r} = (1+t+t^2+\dots)^r = 1 + \binom{r}{1}t + \underbrace{\left(\binom{r}{1}+\binom{r}{2}\right)}_{\binom{r+1}{2}}t^2 + \dots$$

Hilbert-Serre Theorem

$R = \oplus R_n$ $\quad M = \oplus M_n$

$R_0 = k$ (Artinian)

R: f.g./R_0 $\quad M$: f.g./R

$$\Rightarrow H(M,t) = \frac{e(t)}{t^\ell(1-t)^{k_1}\cdots(1-t^{k_r})}$$

where $e(t) \in \mathbb{Z}[t]$, $\ell \geq 0$

and $k_1, \dots, k_r \geq 1$.

Pf. $R = k[x_1, \dots, x_r]$, $x_i \in R_{k_i}$

$$0 \to K \to M(-k_1) \xrightarrow{\cdot x_1} M \longrightarrow L \to 0$$

Notice. x_1 acts as 0 on K and L,

modules over $k[x_2, \dots, x_r]$. By induction:

$$H(K,t) - t^{k_1}H(M,t) + H(M,t) - H(L,t) = 0$$

$$(1-t^{k_1})H(M,t) = H(L,t) - H(K,t) = \frac{e_1(t)}{t^{\ell_1}(1-t^{k_2})\cdots(1-t^{k_r})} - \frac{e_2(t)}{t^{\ell_2}(1-t^{k_2})\cdots(\ )}$$

$$= \frac{e(t)}{t^\ell(1-t^{k_2})\cdots(1-t^{k_r})}.$$

In addition, if $r=0$, then $R=R_0$ $\quad M=\langle m_1,\dots,m_s\rangle$

$$H(M,t)=\sum_{i=1}^{s} t^{l_i}=t^{l_0}\Big(\underbrace{\sum_{i=1}^{s} t^{l_i-l_0}}_{\text{polynomial}}\Big) \qquad \deg m_i = l_i$$

$\square$

Special case: $\deg x_i = 1$, i.e. $x_i\in R_1$ $\forall i=1,2,\dots,r$

$$\Rightarrow H(M,t)=\frac{e(t)}{t^l(1-t)^r}$$

$\exists\, h(M,n)\in\mathbb{Q}[n]$, $\deg h(M,n)=d-1$, leading coef $\dfrac{e(1)}{(d-1)!}$
polynomial

and $\ell(M_n)=h(M,n)$ $\forall\, n\geq \deg(e(t))-l$.

Pf $\quad H(M,t)=e(t)\dfrac{1}{t^l}\displaystyle\sum_{n=0}^{\infty}\binom{d-1+n}{d-1}t^n$

$$\ell(M_n)=e(t)\binom{d-1+n}{d-1}$$

$$=\frac{(n+d-1)(n+d-2)\cdots(n+d-\overbrace{(d-1)}^{=n+1})}{(d-1)!}\cdot e(t)$$

$$=\frac{e(t)}{(d-1)!}n^{d-1}+\dots\text{l.d.t.} \qquad \text{for high degree.}$$

Ex. $\overline{R} = R[x,y,z,w] / (x^2z^2,\ x^2w^2 + xy^2w,\ z^2w^2 - zy^2w)$

$\cap (w=1)$ gives R

$\binom{4-1+n}{3} = \binom{n+3}{3}$

n	0	1	2	3	4	5	6
$\ell(R_n)$	1	4	10	20	35−3	56−12	84−30+2

$\deg = 3$: $x^3, y^3, z^3, w^3, x^2y, x^2z, x^2w, xy^2, xyz, xyw, xz^2, xzw, xw^2$
$y^2z, y^2w, yz^2, yzw, yw^2, z^2w, zw^2$

$$H(R,t) = \frac{e(t)}{t^\ell (1-t)^4} = \frac{e(t)}{t^\ell}\left(1 + 4t + 10t^2 + 20t^3 + 35t^4 + \dots\right)$$

$$\left(\frac{e(t)}{t^\ell} = a_0 + a_1 t + a_2 t^2 + \dots\right)$$

$$= a_0 + (4a_0 + a_1)t + (10a_0 + 4a_1 + a_2)t^2$$
$$+ (20a_0 + 10a_1 + 4a_2 + a_3)t^3 + (35a_0 + 20a_1 + 10a_2 + 4a_3 + a_4)t^4$$
$$+ (56a_0 + 35a_1 + 20a_2 + 10a_3 + 4a_4 + a_5)t^5 + \dots$$
$$= 1 + 4t + 10t^2 + 20t^3 + 32t^4 + 44t^5 + 56t^6 + \dots$$

$\Rightarrow \quad a_0 = 1 \quad a_1 = a_2 = a_3 = 0 \quad a_4 = -3 \quad a_5 = 0 \quad a_6 = 2$

$$H(R,t) = \frac{2t^4 + 4t^3 + 3t^2 + 2t + 1}{(1-t)^2} = \sum_{m=0}^{\infty}\left(\sum_{i=0}^{m} \ell(R_i)\right)t^m$$