Quasi-coherent modules

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When you study commutative algebra may be you are primarily interested in the rings, but probably you start turning your interest towards the modules pretty quickly; they are an important part of the world of rings, and to get the results one wants, one can hardly do without them. The category Mod_A of A-modules is a fundamental invariant of a ring A; and in fact, is the principal object of study in commutative algebra. So also with schemes; for which the so called quasi-coherent $\mathcal{O}_x$-modules form an important attribute of the scheme, if not a decisive part of the structure. They form a category QCoh_X with many properties parallelling those of the category Mod_A—an in case the scheme$X$ is affine, *i.e.,* $X = \operatorname{Spec} A$, they are equivalent. Imposing finiteness conditions on the $\mathcal{O}_X$-modules one arrives at the catergory Coh_X of so called coherent $\mathcal{O}_X$-modules that in the noetherian case parallel the finitely generated A-modules.

We start out by describing the much broader concept of an $\mathcal{O}_X$-module. It is done for schemes, but the concept is meaningfull for any ringed spaced.

In the literature one finds different approaches to the quasi-coherent sheaves. We follow Hartshorn and introduce the quasi-coherent module first for affine schemes. If $X = \operatorname{Spec} A$ one defines an $\mathcal{O}_X$-module $\widetilde{M}$ associated with an A-module M, and in the general case these modules serve as the local models for the quasi-coherent ones. There is a notion of quasi-coherence for $\mathcal{O}_X$-modules on a general locally ringed space. In some other branches of mathematics they are important, *e.g.,* analytic geometry, but we concentrate our efforts on schemes.

Sheaves of modules

A module over a ring is just an additive abelian group equipped with a multiplicative action of A. Loosely speaking we can multiply members of the module by elements from the ring, and of course, the well known series of axioms must be satisfied. In a

similar way, if X is a scheme, an $\mathcal{O}_X$-module is an abelian sheaf F whose sections can be multiplied by sections of $\mathcal{O}_X$; the multiplicator and the multiplicand of course being sections over the same open subset.

Formally, an *$\mathcal{O}_X$-module structure* on the abelian sheaf F is defined as a family of multiplication maps $\Gamma(U,F)\times\Gamma(U,\mathcal{O}_X)\to\Gamma(U,F)$—one for each open subset U of X—making the space of sections $\Gamma(U,F)$ into a $\Gamma(U,\mathcal{O}_X)$-module in a way compatible with all restrictions. That is, for every pair of open subsets $V\subseteq U$, the natural diagram below—where vertical arrows are made up of appropriate restrictions, and the horizontal arrows are multiplications—commutes

$$\begin{tikzcd}
\Gamma(U,F)\times\Gamma(U,\mathcal{O}_X) \arrow[r] \arrow[d] & \Gamma(U,F) \arrow[d] \\
\Gamma(V,F)\times\Gamma(V,\mathcal{O}_X) \arrow[r] & \Gamma(V,F).
\end{tikzcd}$$

Maps, or *homomorphism, of $\mathcal{O}_X$-modules* are just maps $\alpha\colon F\to G$ between $\mathcal{O}_X$-modules considered as abelian sheaves respecting the multiplication by sections of $\mathcal{O}_X$. That is, for any open U the map $\alpha_U\colon\Gamma(U,F)\to\Gamma(U,G)$ is a $\Gamma(U,\mathcal{O}_X)$-homomorphism.

We have now explained what $\mathcal{O}_X$-modules are and told what their homomorphisms should be, and this organizes the $\mathcal{O}_X$-modules into a category that we denote by Mod_X.

The category Mod_X is an *additive category*: The sum of two $\mathcal{O}_X$-homomorphisms as maps of abelian sheaves is again an $\mathcal{O}_X$-homomorphism. So for all F and G the set $\mathrm{Hom}_{\mathcal{O}_X}(F,G)$ is an abelian groups, and the compositions maps are bilinear. Moreover, the direct sum of two $\mathcal{O}_X$-modules as abelian sheaves has an obvious $\mathcal{O}_X$-structure—multiplication being defined componentwise—and is as well the direct sum in the category Mod_X. Infact, this argument works for arbitary direct sums (or coproducts as they also are called). For any family $\{F_i\}$ of $\mathcal{O}_X$-modules, $\bigoplus_{i\in I}F_i$ is an $\mathcal{O}_X$-module(see exercise 0.4 below).

The notions of kernels, cokernels and images of $\mathcal{O}_X$-module homomorphisms now appear naturally. All the three corresponding abelian constructs are invariant under multiplication by sections of $\mathcal{O}_X$, and therefore they have $\mathcal{O}_X$-module structures. The respective defining universal properties (in the category of $\mathcal{O}_X$-modules) come for free, and one easily checks that this makes Mod_X an abelian category; *i.e.,* the kernels of the cokernels equal the cokernels of the kernels.

There is *tensor product* of $\mathcal{O}_X$-modules denoted by $F\otimes_{\mathcal{O}_X}G$. As in many other cases, the tensor product $F\otimes_{\mathcal{O}_X}G$ is defined by first describing a presheaf that subsequently is sheafified. The sections of the presheaf, temporarily denoted by $F\otimes'_{\mathcal{O}_X}G$, are defined in the natural way by

$$\Gamma(U,F\otimes'_{\mathcal{O}_X}G)=\Gamma(U,F)\otimes_{\Gamma(U,\mathcal{O}_X)}\Gamma(U,G).$$

There is also a *sheaf of $\mathcal{O}_X$-homomorphisms* between F and G. The definition goes along lines slightly different from the definition of the tensor product. Recall the sheaf $Hom(F,G)$ of homomorphisms between the abelian sheaves F and G whose

sections over an open set U is the group $\mathrm{Hom}(F|_U, G|_U)$ of homomorphisms between the restrictions $F|_U$ and $G|_U$. Inside this group one has the subgroup of the maps being $\mathcal{O}_X$-homomorphisms, and these subgroups, for different open sets U, are respected by the restriction map. So they form the sections of a presheaf, that turns out to be a sheaf, and that is the sheaf $Hom_{\mathcal{O}_X}(F, G)$ of $\mathcal{O}_X$-homomorphisms from F to G.

PROBLEM 0.1. Assume that F and G are $\mathcal{O}_X$-modules and that $\alpha\colon F \to G$ is a map between them. Show that the kernel, cokernel and image of α as a map of abelian sheaves indeed are $\mathcal{O}_X$-modules, and that they respectively are the kernel, cokernel and image of α in the category of $\mathcal{O}_X$-modules as well. Show that a complex of $\mathcal{O}_X$-modules is exact if and only it is exact as a complex of abelian sheaves. ✷

PROBLEM 0.2. Let F and G be two $\mathcal{O}_X$-modules on the scheme X. Show that the stalk $(F \otimes_{\mathcal{O}_X} G)_x$ at the point $x \in X$ is naturally isomorphic to the tensor product $F_x \otimes_{\mathcal{O}_{X,x}} G_x$ of the stalks F_x and G_x. Show that the tensor product is right exact in the category of $\mathcal{O}_X$-modules. ✷

PROBLEM 0.3. Show that the sheaf-hom $Hom_{\mathcal{O}_X}(F, G)$ of two $\mathcal{O}_X$-modules as defined above is a sheaf. Show that the stalk at a point $x \in X$ of the sheaf-hom $Hom_{\mathcal{O}_X}(F, G)$ equals $\mathrm{Hom}_{\mathcal{O}_{X,x}}(F_x, G_x)$. Show that $Hom_{\mathcal{O}_X}(F, G)$ is right exact in the second variable and left exact in the first. ✷

PROBLEM 0.4. Show that the category Mod_X has arbitrary products and coproducts, by showing that the products and coproducts in the category of abelian sheaves Sh_X are $\mathcal{O}_X$-modules and are the products, respectively the coproducts, in the category Mod_X. ✷

EXAMPLE 0.1. — A BUNCH OF WILD EXAMPLES. The $\mathcal{O}_X$-modules (or at least some of them) play a leading leading role in the theory of schemes, and shortly we shall see a long series of examples. These will all be so called quasi-coherent sheaves. The examples we now describe are a bunch of wild examples, intended to show that $\mathcal{O}_X$-modules without any restrictive hypothesis are very general and often unmanageable objects.

Recall the Godement construction from Notes1. Given any collection of abelian groups $\{A_x\}_{x\in X}$ indexed by the points x of X. We defined a sheaf $\mathcal{A}$ whose sections over an open subset U was $\prod_{x\in U} A_x$, and whose restriction maps to smaller open subsets were just the projections onto the corresponding smaller products. Requiring that each A_x be a module over the stalk $\mathcal{O}_{X,x}$ makes $\mathcal{A}$ into an $\mathcal{O}_X$-module; indeed, the space of sections $\Gamma(U, \mathcal{A}) = \prod_{x\in U} A_x$ is automatically an $\Gamma(U, \mathcal{O}_X)$-module, the multiplication being defined componentwise with the help of the stalk maps $\Gamma(U, \mathcal{O}_X) \to \mathcal{O}_{X,x}$. Clearly this module structures is compatible with the projections, and thus makes $\mathcal{A}$ into an $\mathcal{O}_X$-module. ❄

EXAMPLE 0.2. — MODULES ON SPECTRA OF DVR'S. Modules on the prime spectrum of a discrete valuation ring R are particularly easy to describe. Let K denote the fraction field of R. The scheme $X = \operatorname{Spec} R$ has only two non-empty open sets, the whole space X it self and the singleton $\{\eta\}$ where η denotes the generic point. The singleton $\{\eta\}$ is underlying set of the open subscheme $\operatorname{Spec} K$. The data of an $\mathcal{O}_X$-module F consist of an R module M, that is the space $\Gamma(X, F)$ of global sections of F, and a vector space N over K, being the space of sections $\Gamma(\{\eta\}, F)$. The restriction map can be just any R-module homomorphism $M \to N$.

This homomorphism can very well be zero, and in that case M and N can be completely arbitrary modules. Again this illustrates the versatility of general $\mathcal{O}_X$-modules. ❄

PROBLEM 0.5. Assume that $p_1, \ldots, p_r$ is a set of primes, and let $\mathbb{Z}_{(p_i)}$ as usual denote the localization at prime ideal (p_i). Let X be the scheme obtained by glueing the schemes $X_i = \operatorname{Spec} \mathbb{Z}_{(p_i)}$ together along their common open subschemes $\operatorname{Spec} \mathbb{Q}$. Describe the $\mathcal{O}_X$-modules on X. ✸

PROBLEM 0.6. Let $A = \prod_{1 \le i \le n} K_i$ be the product of finitely many fields. Describe the category Mod_X. ✸

Quasi-coherent sheaves

In The Oxford English Dictionary there several nuances of the word coherent are given, but the one at top is:
That sticks or clings firmly together; esp. united by the force of cohesion. Said of a substance, material, or mass, as well as of separate parts, atoms, etc.

So the *coherence* of a sheaf should mean that there are some strong relations between the sections over different open sets, at least over sets from some sufficiently large collection of open sets. In our context the open affine subsets stand out as obvious candidates to form such a collection, and indeed, a *quasi-coherent* sheaf on the scheme X will turn out[1] to have the following coherence property: If $U \subseteq V$ are two affine open sets in X there is a canonical map

$$\Gamma(U, F) \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(V, \mathcal{O}_V) \to \Gamma(V, F) \tag{1}$$

sending $s \otimes f$ to $\rho_V^U(s) \cdot f$, where ρ_V^U as usual indicates the restriction maps in $\mathcal{O}_X$. The salient point is that for a *quasi-coherent* sheaf F this map is an *isomorphism*. In fact the converse holds true as well, so the map in (1) being an isomorphism for all pairs $V \subseteq U$ of open affine sets is equivalent to F being quasi-coherent.

[1] One might have used this coherence property as the definition, but because of obscure reasons we choose another definition.

To pin down the quasi-coherent sheaves, one first establishes a collection of model-sheaves on affine schemes. To each A-module M one associates an $\mathcal{O}_X$-module $\widetilde{M}$ on $X = \operatorname{Spec} A$, and on a general scheme X, a quasi-coherent module F is going to be one that locally is of the form $\widetilde{M}$.

The *coherent modules* are the quasi-coherent modules that satisfy a certain finiteness condition that in the noetherian case boils down to the $\mathcal{O}_X$-module being finitely generated. These coherent $\mathcal{O}_X$-modules are the $\mathcal{O}_X$-modules most frequently encountered in algebraic geometry and hence they have gotten the shortest name, even though it is being quasi-coherent that means satisfying a coherence property.

COHERENT MODULES OVER A RING Let A be a ring and let M be an A-module. The module M is *of finite presentation* if for some integers n and m there is an exact sequence

$$A^n \longrightarrow A^m \longrightarrow M \longrightarrow 0\,.$$

One says that M is *coherent* if the following two requirements are fulfilled

- ☐ M is finitely generated.
- ☐ The kernel of every surjection $A^n \to M$ is finitely generated.

Being coherent is the strongest of the the three properties, next comes being of finite presentation and being finitely generated is the weakest. However, in the case A is a noetherian ring—which frequently is the case in algebraic geometry—a module M being coherent is equivalent to M being finitely generated. The notion of coherence comes from the theory analytic functions where coherent non-noetherian rings are frequent.

As an example of what coherence entails, we mention that the ring A itself is a coherent A-module if and only if all finitely generated ideals are of finite presentation.

PROBLEM 0.7. Show that last statement about A being coherent over itself ✸

PROBLEM 0.8. Show that if A is a noetherian ring, then the three conditions of *coherence*, *finitely generation* and *being of finite presentation* on an A-module M coincide. ✸

Quasi-coherent sheaves on affine schemes

For each A-module M we shall exhibit an $\mathcal{O}_X$-module $\widetilde{M}$; the construction of which completely parallels what we did when constructing the structure sheaf $\mathcal{O}_X$ on $X = \operatorname{Spec} A$. One defines a presheaf on X, temporarily denoted by M', simply by letting the sections over an open U be

$$\Gamma(U, M') = M \otimes_A \Gamma(U; \mathcal{O}_X). \tag{2}$$

The restriction maps of M' are induced by the restriction maps of the structure sheaf

$\mathcal{O}_X$. That is, if $U \subseteq V$ are two open subsets and $\rho_V^U \colon \Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_X)$ the restriction map of the structure sheaf, the restrictions of M' are simply be the maps $\mathrm{id}_M \otimes \rho_V^U$. This clearly yields an $\mathcal{O}_X$-module, multiplication being performed in the second factor of the tensor product. *A priori* the $\mathcal{O}_X$-module M' is only a presheaf, so we let $\widetilde{M}$ be the sheafification.

For any A-module homomorphism $\alpha \colon M \to N$ there is an obvious way of obtaining an $\mathcal{O}_X$-module homomorphism $\tilde{\alpha} \colon \widetilde{M} \to \widetilde{N}$; indeed, the maps $\alpha \otimes \mathrm{id}_{\Gamma(U;\mathcal{O}_X)}$ are $\Gamma(U, \mathcal{O}_X)$-homomorphisms from $\Gamma(U, M')$ to $\Gamma(U, N')$ compatibel with the restrictions, and thus induce a map between M' and N'. The map $\tilde{\alpha}$ is the associated map between the sheafifications. Clearly one has $\widetilde{\phi \circ \psi} = \widetilde{\phi} \circ \widetilde{\psi}$, and the "tilde-operation" is therefore a functor $\mathsf{Mod}_A \to \mathsf{Mod}_{\mathcal{O}_X}$.

The three main properties of $\widetilde{M}$ are listed below. They are completly analogous to the statements in proposition **??** on page **??** in Notes2 about the structure sheaf $\mathcal{O}_X$; and as well, the proofs are *mutatis mutandis* the same.

- Stalks: Let $x \in \operatorname{Spec} A$ be a point whose corresponding prime ideal is $\mathfrak{p}$. The stalk $\widetilde{M}_x$ of M at $x \in X$ is $\widetilde{M}_x = M_{\mathfrak{p}} = M \otimes_A A_{\mathfrak{p}}$.
- Sections over distinguished open sets: If $f \in A$ one has $\Gamma(D(f), \widetilde{M}) = M_f = M \otimes_A A_f$. In particular it holds true that $\Gamma(X, \widetilde{M}) = M$.
- Sections over arbitraty open sets. For any open subset U of $\operatorname{Spec} A$ covered by the distinguished sets $\{D(f_i)\}_{i \in I}$, there is an exact sequence

$$0 \longrightarrow \Gamma(U, \widetilde{M}) \xrightarrow{\alpha} \textstyle\prod_i M_{f_i} \xrightarrow{\rho} \prod_{i,j} M_{f_i f_j} \ . \tag{3}$$

Only the second of the three properties needs a separate proof, the two others follow formally. The first follows since both the stalks M_x and the localizations $M_{\mathfrak{p}}$ are direct limits of the same modules over the same inductive system (indexed by the distinguished open subsets $D(f)$ containing x), and the third is just the general exact sequence expressing the space of sections of a sheaf over an open set in terms of the space of sections over members of an open covering.

Lemma 0.1 *For any two A-modules M and N, the association $\phi \to \tilde{\phi}$ is an isomorphism* $\mathrm{Hom}_A(M, N) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$.

In the lingo of category theory the lemma is expressed by saying that the tilde-operation is a *fully faithful functor* (trofast full funktor), *fully* meaning the map in the lemma is surjective and *faithful* that it is injective. Loosely speaking, the "tilde-operation" gives an isomorphism of Mod_A with a subcategory of $\mathsf{Mod}_{\mathcal{O}_X}$. It is a strict subcategory; most of the $\mathcal{O}_X$-modules are not of the form $\widetilde{M}$, but those that are, form the category of quasi-coherent $\mathcal{O}_X$-modules, that we shortly return to.

EXAMPLE 0.3. The example of an discrete valuation ring is always useful to consider, and we continue exploring example 0.2 above. The $\mathcal{O}_X$ module given by the data $M \to N$ is of the form $\widetilde{M}$ if and only if $N = M \otimes_R K$ and the restriction map is the canonical map $M \to M \otimes_R K$ sending m to $m \otimes 1$. ❄

Assume that an $\mathcal{O}_X$-module F is given on $X = \operatorname{Spec} A$, and let M denote the global sections of F; that is, M is the A-module $M = \Gamma(X, F)$. There is a natural map $\widetilde{M} \to F$ of $\mathcal{O}_X$-modules. By an appropriate glueing lemma—the one for maps between sheaves—it suffices to tell what the map does to sections over members of a basis for the topology, *e.g.,* over open distinguished sets. The sheaf F being an $\mathcal{O}_X$-module, multiplication by f^{-1} in the space of sections $\Gamma(D(f), F)$ has a meaning since $\Gamma(D(f), \mathcal{O}_X) = A_f$. Hence we may send the section $mf^{-n} \in M_f$ of $\widetilde{M}$ to the section of F over $D(f)$ obtained by multiplying the restriction of m to $D(f)$ by f^{-n}; *i.e.,* we send m to $f^{-n} \cdot m|_{D(f)}$. For later reference we state this observation as a lemma, leaving the task of checking the details to the zealous student:

Lemma 0.2 *Given a quasi-coherent sheaf F on the affine scheme $X = \operatorname{Spec} A$. Then there is a unique $\mathcal{O}_X$-module homomorphism $\theta_F \colon \Gamma(X, F)\tilde{} \to F$ inducing the identity on the spaces of global sections.*

Moreover, it is natural in the sense that if $\alpha \colon F \to G$ is a map of $\mathcal{O}_X$-module inducing the map $a \colon \Gamma(X, F) \to \Gamma(X, G)$ on global sections, one has $\theta_G \circ \tilde{a} = \alpha \circ \theta_F$.

PROBLEM 0.9. Check that this is a well defined map (there is a choice involved in the definition). ✸

Lemma 0.3 *In the canonical identification of the distinguished open subset $D(f)$ with $\operatorname{Spec} A_f$, the $\mathcal{O}_X$-module $\widetilde{M}$ restricts to $\widetilde{M_f}$.*

PROOF: As $\Gamma(D(f), \widetilde{M}) = M_f$, by the comment preceding the lemma, there is a map $\widetilde{M_f} \to \widetilde{M}|_{D(f)}$ that on distinguished open subsets $D(g) \subseteq D(f)$ induces an isomorphism between the two spaces of sections, both being equal to the localization M_g. ❑

The "tilde-functor" is really a no-nonsense functor having almost all properties one can desire. It is fully faithful, as we have seen, but it also is an additive as well as an exact functor. Moreover, it takes the tensor product $M \otimes_A N$ of two A-modules to the tensor product $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ of the corresponding $\mathcal{O}_X$-modules, and if M is of finite presentations, the A-module of homomorphisms $\operatorname{Hom}_A(M, N)$ to the sheaf of homomorphisms $\mathit{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$. However, this is nor true if M is not of finite presentation, the only lacking desirable property of the functor $\widetilde{(-)}$.

To verify the first of these claims, assume given an exact sequence of A-modules:

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

That the induced sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \widetilde{M'} \longrightarrow \widetilde{M} \longrightarrow \widetilde{M''} \longrightarrow 0$$

is exact is a direct consequence of the three following facts. The stalk of a tilde-module $\widetilde{M}$ at the point x with corresponding prime ideal $\mathfrak{p}$ is $M_{\mathfrak{p}}$, localization is an exact functor, and finally, a sequence of abelian sheaves is exact if and only if the sequence of stalks at every point is exact.

For the tensor product, observe that by the definition of the tensor product of two $\mathcal{O}_X$-modules, there is a canonical map $M \otimes_A N \to \Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})$, and by the comment preceding lemma 0.3 above it induces a map $(M \otimes_a N)^{\sim} \to \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$. This turns out to be an isomorphisms since stalk by stalk it is an isomorphisms (the stalk of the tensor product being the tensor product of the stalks, you did exercise 0.2 didn't you?).

When it comes to hom's however, the situation is somehow more subtle. If M is not of finite presentation, it is not true that $\mathrm{Hom}_A(M, N)$ localizes. There is always a canonical and obvious map

$$\mathrm{Hom}_A(M, N) \otimes_A A_f \to \mathrm{Hom}_{A_f}(M_f, N_f), \tag{4}$$

that sends α to $\alpha \otimes \mathrm{id}_{A_f}$, but without some finiteness condition (like being of finite presentation) on M, it is not necessarily an isomorphism. Even in the simplest case of a infinitely generated free module $M = \bigoplus_{i \in I} Ae_i$ that map is not surjective. An element in $\mathrm{Hom}_A(M, N) \otimes_A A_f$ is of the form $\alpha \otimes f^{-n}$. An element in $\mathrm{Hom}_{A_f}(M_f, N_f)$, however, is given by its values $m_i \otimes f^{-n_i}$ on the free generators e_i, and the salient point is that the n_i's may tend to infinity, and no n working for all i's can be found.

PROBLEM 0.10. Show that the map (4) above is an isomorphism when M is of finite presentation. HINT: First observe that this holds when $M = A$. Then use a presentation of M to reduce to that case. ✸

If M is of finite presentations, one has The global sections of the sheaf-hom $\mathit{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$ equals $\mathrm{Hom}_A(M, N)$, and there is a map

$$\mathrm{Hom}_A(M, N)^{\sim} \to \mathit{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}).$$

In case M is of finite presentation, the maps in (4) are isomorphisms, and the map above induces isomorphisms between the spaces of sections of the two sides over any distinguished open subset. One concludes that the map is an isomorphism, and one has $\mathrm{Hom}_A(M, N)^{\sim} \simeq \mathit{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$.

In short, we have established the important proposition:

Proposition 0.1 *Assume that A is a ring and let $X = \mathrm{Spec}\, A$. The functor from the category Mod_A of A-modules to the category $\mathsf{Mod}_{\mathcal{O}_X}$ of $\mathcal{O}_X$-modules given by $M \to \widetilde{M}$ enjoys the following three properties*

- ☐ *It is a fully faithful additive and exact functor.*
- ☐ *One has a canonical isomorphism* $(M \otimes_A N)\widetilde{} \simeq \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$.
- ☐ *If M is of finite presentation, one has a canonical isomorphism* $\mathrm{Hom}_A(M,N)\widetilde{} \simeq \mathit{Hom}_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})$.

PROBLEM 0.11. Show that the tilde-functor is additive; *i.e.,* takes direct sums of modules to the direct sum and sums of maps to the corresponding sums. ✸

EXAMPLE 0.4. Assume that A is an integral domain and that K is the field of fractions of A. Show that the $\mathcal{O}_X$-module $\widetilde{K}$ is a constant sheaf in the strong sense, that is $\Gamma(U,\widetilde{K}) = K$ for any non empty open $U \subseteq X$ and the restriction maps all equal the identity id_K. Hence $\widetilde{K}$ is a *flabby sheaf.* ❄

FUNCTOTIALITY The setting is that of a map between the two affine schemes X and Y; say $\phi\colon X \to Y$. We let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ and we let $\phi^\sharp\colon B \to A$ be the map corresponding to ϕ.

If M is an A-module, can one describe the sheaf $\phi_*\widetilde{M}$ on Y? The answere is not only yes but the description is very simple. The A-module M can be considered a B-module via the map $B \to A$ and as such is denoted by M_B. Clealy this is a functorial construction in M. In this setting one has

Proposition 0.2 *One has* $\phi_*\widetilde{M} = \widetilde{M_B}$.

PROOF: There is an obvious map $\widetilde{M_B} \to \phi_*(\widetilde{M})$ as $\Gamma(Y,\phi_*(\widetilde{M})) = \Gamma(X,\widetilde{M}) = M$. The argument that follows shows it is an isomorphism, and it is as usual sufficient to verify that sections over distinguished open sets of the two sides coincide. The crucial observation is that $\phi^{-1}D(f) = D(\phi^\sharp f)$—indeed, a prime ideal $\mathfrak{p} \subseteq A$ satisfies $f \in \phi^{\sharp\,-1}(\mathfrak{p})$ if and only if $\phi^\sharp(f) \in \mathfrak{p}$. As f acts on M_B as multiplication by $\phi^\sharp(f)$, one clearly has $(M_B)_f = M_{\phi^\sharp(f)} = \Gamma(D(\phi^\sharp(f)), M)$. ❑

Quasi-coherence sheaves on general schemes

Having established the sheaves on affine space that serve as local models for the quasi-coherent sheaves, we are now ready for the general definition. If X is a scheme and F an $\mathcal{O}_X$-module, one says that F is *quasi-coherent* (kvasikoherent) $\mathcal{O}_X$-module, or *quasi-coherent sheaf* for short, if there is an open affine covering $\{U_i\}_{i\in I}$ of X, say $U_i = \operatorname{Spec} A_i$, and modules M_i over A_i such that $F|_{U_i} \simeq \widetilde{M_i}$. Phrased in slightly different manner, $\mathcal{O}_X$-module F is quasi-coherent if the restriction $F|_{U_i}$ of F to each U_i is of type tilde of an A_i-module. In particular, the modules $\widetilde{M}$ on affine schemes $\operatorname{Spec} A$ are all quasi-coherent.

The restriction of a quasi-coherent sheaf F to any open set $U \subseteq X$ is quasi-coherent. Indeed, it will suffices to verify this for X an affine scheme, and by lemma 0.3 the restriction of a sheaf of tilde-type to a distinguished open set is of tilde-type. As any

open U in an affine scheme is the union of distinguished open subsets, it follows that $F|_U$ is quasi-coherent.

For F to be quasi-coherent, we require that F be locally of tilde-type for just *one* open affine cover. However, it turns out that this will hold for *any* open affine cover, or equivalently, that $F|_U$ is of tilde-type for any open affine subset $U \subseteq X$. This is a much stronger than the requirement in the definition, and it is somehow subtle to prove. As a first corollary we arrive at the *a priori* not obvious conclusion that the modules of the form $\widetilde{M}$ are the only quasi-coherent $\mathcal{O}_X$-modules on an affine scheme. We shall also see that quasi-coherent modules enjoy the coherence property (1) on page 4 that was the point of departure for our discussion.

The story starts with a lemma that establishes the coherence property (1) in a very particular case; *i.e.,* for sections over distinguished open sets of a quasi-coherent $\mathcal{O}_X$-module on an affine scheme $X = \operatorname{Spec} A$. For any distinguished open set $D(f) \subseteq X$ it holds that $\Gamma(D(f), \mathcal{O}_X) = A_f$, and consequently there is for any $\mathcal{O}_X$-module a canonical map $\Gamma(X, F) \otimes_A A_f \to \Gamma(D(f), F)$ sending $s \otimes af^{-n}$ to $af^{-n} \cdot s|_{D(f)}$. It turns out to be an isomorphism whenever F is quasi-coherent:

Lemma 0.4 *Suppose that $X = \operatorname{Spec} A$ is an affine scheme and that F is a quai-coherent $\mathcal{O}_X$-module. Let $D(f) \subseteq X$ be a distinguished open set. Then one has*

- $\Gamma(D(f), F) \simeq \Gamma(X, F) \otimes_A A_f$.
- *Let $s \in \Gamma(X, F)$ be a global section of F and assume that $s|_{D(f)} = 0$, then sufficiently big powers of f kill s, that is, for sufficiently big integers n one has $f^n s = 0$.*
- *Let $s \in \Gamma(D(f), F)$ be a section. Then for a sufficiently large n, the section $f^n s$ extends to a global section of F. That is, there exists an n and a global section $t \in \Gamma(X, F)$ such that $t|_{D(f)} = f^n s$*

PROOF: The first statement is by the definition of localization equivalent to the two others.

The sheaf F being quasi-coherent by hypothesis, and the affine scheme $X = \operatorname{Spec} A$ being quasi-compact, there is a finite open affine covering of X by distinguished sets $D(g_i)$ such that $F|_{D(g_i)} \simeq \widetilde{M_i}$ for some A_{g_i}-modules M_i. The section s of F restricts to sections s_i of $F|_{D(g_i)}$ over $D(g_i)$, that is, to elements s_i of M_i.

Further restricting F to the intersections $D(f) \cap D(g_i) = D(fg_i)$ yields the equality $F|_{D(fg_i)} = (\widetilde{M_i})_f$, and by hypothesis, the section s restricts to zero in $\Gamma(D(fg_i), F) = (M_i)_f$. This means that the localization map sends s_i to zero in $(M_i)_f$. Hence s_i is killed by some power of f, and since there is only finitely many g_i's, there is an n with $f^n s_i = 0$ for all i; that is, $(f^n s)|_{D(g_i)} = 0$ for all i. By the locality axiom for sheaves, it follows that $f^n s = 0$.

Assume now a section $s \in \Gamma(D(f), F)$ is given. We are to see that $f^n s$ extends to a global section of F for large n. Each restriction $s|_{D(fg_i)} \in \Gamma(D(fg_i), F) = (M_i)_f$ is

of the form $f^{-n}s_i$ with $s_i \in M_i = \Gamma(D(g_i), F)$, and by the usual finiteness argument, n can be chosen uniformly for all i. This means that $s_i = f^n s$ and $s_j = f^n s$ mach on the intersection $D(f) \cap D(g_i) \cap D(g_j)$, and by the first part of the lemma applied to $\operatorname{Spec} A_{g_i g_j}$, one has $f^N(s_i - s_j) = 0$ on $D(g_i) \cap D(g_j)$ for a sufficiently large integers N. Hence the different $f^N s_i$'s patch together to give the desired global section t of F. ❑

Theorem 0.1 *Let X be a scheme and F an $\mathcal{O}_X$-module. Then F is quasi-coherent if and only if for all open affine subsets $U \subseteq X = \operatorname{Spec} A$, the restriction $F|_U$ is isomorphic to a an $\mathcal{O}_X$-module of the form $\widetilde{M}$ for an A-module M.*

PROOF: As quasi-coherence is conserved when restricting $\mathcal{O}_X$-modules to open sets, we may surely assume that X itself is affine; say $X = \operatorname{Spec} A$. Let $M = \Gamma(X, F)$. We saw in lemma 0.2 on page 7 that there is a natural map $\widetilde{M} \to F$ that on distinguished open sets sends mf^{-n} to $f^{-n}m|_U$, but by the fundamental lemma 0.4 above, this is an isomorphism between the spaces of sections over the distinguished open sets. Hence the two sheaves are isomorphic. ❑

Applying this to an affine scheme, yields the important fact that any quasi-coherent sheaf F on an affine scheme $X = \operatorname{Spec} A$ is of the form $\widetilde{M}$ for an A-module M.

Proposition 0.3 *Assume that $X = \operatorname{Spec} A$. The tilde-functor $M \mapsto \widetilde{M}$ is an equivalence of categories Mod_A and QCoh_X with the global section functor as an inverse. Finitely presented modules correspond to coherent sheaves. In particular, if A is noetherian, finitely generated modules correspond to coherent sheaves.*

When speaking about mutually inverse functors one should be very careful; in most cases such a statement is an abuse of language. Two functors F and G are mutually inverses when there are natural transformations, both being an isomorphisms, between the compositions $F \circ G$ and $G \circ F$ and the appropriate identity functors. In the present case one really has an *equality* $\Gamma(X, \widetilde{M}) = M$, so that $\Gamma \circ \widetilde{(-)} = \mathrm{id}_{\mathsf{Mod}_A}$. On the other hand, the natural transformation $\Gamma(X, F)\widetilde{\ } \to F$ from lemma 0.2 on page 7 furnishes the required isomorphism of functors.

Coherence

Let A be a ring and let M be an A-module. The module M is *of finite presentation* if for some integers n and m there is an exact sequence

$$A^n \longrightarrow A^m \longrightarrow M \longrightarrow 0\,.$$

One says that M is *coherent* if the following two requirements are fulfilled

- ☐ M is finitely generated.
- ☐ The kernel of every surjection $A^n \to M$ is finitely generated.

The second statement is equivalent to every finitely generated submodule in M being of finite presentation. In the case A is a noetherian ring—which frequently is the case in algebraic geometry—a module M being coherent is equivalent to M being finitely generated. The condition comes from the theory analytic functions where coherent non-noetherian rings are frequent.

On a scheme X a quasi-coherent $\mathcal{O}_X$-module is *coherent* if there is a covering of X by open affine sets $U_i = \operatorname{Spec} A_i$ such that $F|_{U_i} = \widetilde{M_i}$ with the M_i's being coherent A_i-modules.

PROBLEM 0.12. Let X be a scheme and let F be a sheaf of $\mathcal{O}_X$-modules. Shoe that F is quasi-coherent if and only if every point $x \in X$ has an open neigbourhood U over which there are an exact sequences of $\mathcal{O}_X$-modules

$$\mathcal{O}_X^I|_U \longrightarrow \mathcal{O}_X^J|_U \longrightarrow F|_U \longrightarrow 0$$

In the sequnce the exponents I and J are not necessaily finite cardinals. Shpw that F is coherent if and only if one may take I nd J finite in the above sequence. ✸

Theorem 0.2 *Let X be a scheme and let F be an $\mathcal{O}_X$-module on X. Then F is quasi-coherent of and only if for any pair $V \subseteq U$ open affine subsets, the natural map*

$$\Gamma(U, F) \otimes_{\Gamma(U,\mathcal{O}_X)} \Gamma(V, \mathcal{O}_X) \to \Gamma(V, F) \tag{5}$$

is an isomorphism.

PROOF: We may clearly assume that X is affine, say $X = \operatorname{Spec} A$. Assume that the maps (5) are isomorphisms. We may take $V = D(f)$ (and $U = X$) and $M = \Gamma(X, F)$. Then from (5) it follows that $\Gamma(D(f), F) = M_f$ which shows that the canonical map $\widetilde{M} \to F$ is an isomorphism over all distinguished open subsets, and therefore an isomorphism.

To argue for the implication the other way, assume that $U = \operatorname{Spec} B$ and that F is quasi-coherent; that is, $F = \widetilde{M}$ for some A-module M after theorem 0.2. The restriction of a quasi-coherent sheaf is quasi-coherent, so $M|_U = \widetilde{N}$ for some B-module N, and the map in (5) is just a map have map $M \otimes_A B \to N$. It induces an isomorphism over all local rings $A_\mathfrak{p} = B_\mathfrak{p}$ (where $\mathfrak{p} \in U = \operatorname{Spec} B$) since $\widetilde{M}|_U \simeq \widetilde{N}$ and therefore is an isomorphism of B-modules. ❑

Closedness

The following lemma is an easy consequence of the global section functor being exact on the category of quasi-coherent sheaves on an affine scheme once we have the general machinery of cohomology at our disposal. It possible to give an *ad hoc* proof (*e.g.*, see Hartshorn xxx) but a proof is quite subtle and may be not worth while doing

as the conceptual proof using cohomology is clean and straightforward. It is of course also possible to extract exactly what is needed from the general theory to prove the lemma,. . . .

Lemma 0.5 *Assume that $X = \operatorname{Spec} A$ is an affine scheme. Suppose we are given exact sequence*

$$0 \longrightarrow \widetilde{M} \longrightarrow F \longrightarrow G \longrightarrow 0$$

of $\mathcal{O}_X$ modules F, G and $\widetilde{M}$ where $\widetilde{M}$ is quasi-coherent. The the corresponding sequence of global sections

$$0 \longrightarrow M \longrightarrow \Gamma(X, F) \longrightarrow \Gamma(X, G) \longrightarrow 0$$

is exact.

PROOF: Let σ be a global section of G. By Zorn's lemma there is a maximal open set U suct that there exists σ' of F over U mapping to $\sigma|_U$. We are to show that $U = X$, so assume the contrary. Let $x \in X$ and let V be an open distinguisjed neigbpourhood of x such that $\sigma|_V$ can be lifted to a section τ of F over V. Then $t - s$ lies in $\widetilde{M}$ ❑

Proposition 0.4 *Assume that $\alpha\colon F \to G$ is a map of quasi-coherent sheaves on the scheme X. The kernel, cokernel and the image of α are all quasi-coherent. The category QCoh_X is closed under extensions; that is, if*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \tag{6}$$

is a short exact sequence of $\mathcal{O}_X$-modules with the two extreme sheaves M' and M'' being quasi-coherent, the middle sheaf M is quasi-coherent as well.

PROOF: If $\alpha\colon F \to G$ is a map of quasi-coherent $\mathcal{O}_X$-modules, on any open affine subsets $U = \operatorname{Spec} A$ of X it may be described as $\alpha|_U = \widetilde{a}$ where $a\colon M \to N$ is a A-module homomorphism and M and N are A-modules with $F|_U = \widetilde{M}$ and $G|_U = \widetilde{N}$. Since the tilde-functor is exact, one has $\operatorname{Ker} \alpha|_U = (\operatorname{Ker} a)\tilde{}$. Moreover, by the same reasoning, it holds true that $\operatorname{Coker} \alpha|_U = (\operatorname{Coker} a)\tilde{}$ and $\operatorname{Im} \alpha|_U = (\operatorname{Im} a)\tilde{}$.

Suppose now that an extension like (6) is given. The leftmost sheaf M' being quasi-coherent lemma 0.5 entails that the induced sequence of global sections is exact; that is, the upper horizontal sequence in the diagram below. The three vertical maps in the diagram are the natural maps from lemma 0.2 on page 7. Since M' and M'' both are quasi-coherent sheaves, the two flanking vertical maps are isomorphisms, and the

snake lemma implies that the middle vertical map is an isomorphism as well. Hence M is quasi-coherent.

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X,M')\tilde{} \arrow[r] \arrow[d] & \Gamma(X,M)\tilde{} \arrow[r] \arrow[d] & \Gamma(X,M'')\tilde{} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

❑

The category QCoh_X is an abelain category with tensor products and internal hom's.

Functoriality

Theorem 0.3 *Assume that $\phi\colon X \to Y$ is a morphism of schemes and that F is a quasi-coherent sheaf on X. If X is separated and ϕ is quasi-compact, then the direct image $\phi_* F$ is quasi-coherent on Y.*

EXAMPLE 0.5. Let $X = \bigcup_{i\in I} \operatorname{Spec}\mathbb{Z}$ be the disjoint union of countably many copies of $\operatorname{Spec}\mathbb{Z}$ and let $\phi\colon X \to \operatorname{Spec}\mathbb{Z}$ be the morphism that equals the identity on each of the copies of $\operatorname{Spec}\mathbb{Z}$ that constitute X. Then $\phi_*\mathcal{O}_X$ is not quasi-coherent. Indeed, the global sections of $\phi_*\mathcal{O}_X$ satisfy $\Gamma(\operatorname{Spec}\mathbb{Z}, \phi_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = \prod_{i\in I}\mathbb{Z}$. On the other hand if p is any prime, one has $\Gamma(D(p), \phi_*\mathcal{O}_X) = \Gamma(\phi^{-1}D(p), \mathcal{O}_X) = \prod_{i\in I}\mathbb{Z}[p^{-1}]$. It is not true that $\Gamma(D(p), \phi_*\mathcal{O}_X) = \gamma(\operatorname{Spec}\mathbb{Z}, \phi_*\mathcal{O}_X) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$ hence $\phi_*\mathcal{O}_X$ is not quasi-coherent. Indeed, elements in $\prod_{i\in I}\mathbb{Z}[p^{-1}]$ are sequences of the form $(z_i p^{-n_i})_{i\in I}$ where $z_i \in \mathbb{Z}$ and $n_i \in \mathbb{N}$. Such an element lies in $(\prod_{i\in I}) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}])$ only if the n_i's form a bounded sequence, which is not the case for general elements of shape $(z_i p^{-n_i})_{i\in I}$ when I is infinite. ❄

PROOF: Being quasi-coherent is a local property for a sheaf, so we may assume that Y is affine, say $Y = \operatorname{Spec} A$. Then as ϕ is assumed to be quasi-compact, X is quasi-compact and can be covered by finitely many open affine subsets U_i, and s X is assumed to be separated, we know that sets $U_{ij} = U_i \cap U_j$ are affine as well. For any open $V \subseteq U$ one has the usual exact sequence

$$0 \longrightarrow \Gamma(V, \phi_* F) \longrightarrow \textstyle\prod_i \Gamma(U_i \cap \phi^{-1}V, \mathcal{O}_X) \longrightarrow \prod_{i,j} \Gamma(U_{ij} \cap V, \mathcal{O}_X). \tag{7}$$

The sequence is compatible with the restriction maps induced from an inclusion $V' \subseteq V$, hence gives rise to the following exact sequence of sheaves on X:

$$0 \longrightarrow \phi_*\mathcal{O}_X \longrightarrow \textstyle\prod_i \phi_{i*}\mathcal{O}_{U_i} \longrightarrow \prod_{i,j} \phi_{ij*}\mathcal{O}_{U_{ij}} \tag{8}$$

where $\phi_i = \phi|_{U_i}$ and $\phi_{ij} = |_{U_{ij}}$. Indeed, the sequence (7) is nothing but the sequence obtained by taking sections over the open subset V of the sequence (8). Now, each of the sheaves $\phi_{i*}\mathcal{O}_{Ui}$ and $\phi_{ij*}\mathcal{O}_{U_{ij}}$ are quasi-coherent by the affine case of the theorem

(proposition 0.2 on page 9). They are finite in number as the covering U_i is finite. Hence $\prod_i \phi_{i*}\mathcal{O}_{U_i}$ and $\prod_{i,j} \phi_{ij*}\mathcal{O}_{U_{ij}}$ are finite products of quasi-coherent $\mathcal{O}_X$-modules and therefore they are quasi-coherent. The theorem the follows from proposition 0.4 on page 13. ❑

PROBLEM 0.13. Show that $X = \bigcup_{i\in I} \operatorname{Spec}\mathbb{Z}$ is not affine. Show that in the category Sch of schemes X is the coproduct of the infinitely many copies of $\operatorname{Spec}\mathbb{Z}$ involved . Show that $\operatorname{Spec}\prod_i \mathbb{Z}$ is the coproduct of the copies of $\operatorname{Spec}\mathbb{Z}$ in the categiry Aff of affine schemes. Show that these gadgeds are subtatially different. ✸