The Hilbert Schemes of Curves in $\mathbf{P}^3$

by
Yoon-Ho Alex Lee

a thesis
presented to the Department of Mathematics
in partial fulfillment
of the requirements for the degree
of Bachelor of Arts with Honors

Harvard University
Cambridge, Massachusetts
April 3, 2000 (Revised May 9, 2000)

Acknowledgments

Above all, I would like to thank my advisor Professor Joseph Harris. From the first time he taught me algebraic geometry during my junior spring semester until just days before the due date of this thesis, he has exercised unparalleled patience and warmth as a professor and a friend. The inception of this thesis would not have been possible without his landmark paper *Curves in Projective Spaces*. He has taken a large chunk of his valuable time to explain the specific details of his paper numerous times until I reached full understanding. His emotional and psychological support deserves much credit for helping me maintain my focus through the process. Most importantly, his enthusiasm in this subject matter kept me excited about writing this paper.

I would also like to thank Jason Starr for our countless weekly meetings during the spring semester. Mr. Starr spent hours and hours explaining the commutative ring theory and the geometry behind the Hilbert schemes. He never hesitated to go through another example to help me understand the algebra. I enjoyed discussing algebraic geometry with Mr. Starr tremendously. I wish him the best of luck with his upcoming years at MIT as a junior professor.

I am also indebted to the following people: Professor Brian Conrad and Professor Tom Graber of Harvard University, Professor Steve Kleiman from MIT, Professor Michael Schlessinger from University of North Carolina, Chapel Hill, Professor Ragni Piene from University of Oslo, Norway, and Professor Daniel Grayson from University of Illinois at Urbana-Champaign. While my encounters with these people have not been extensive, their contributions to this paper are enormous.

A few of my peers have been especially helpful in allowing me opportunities to discuss mathematics casually with them. They include Jacob Lurie and Josh Nichols-Barrer. Nils Barth provided me with some invaluable technical assistance with LaTeX which made the presentation of this paper possible.

I am grateful to Henry Cohn and Mak Trifkovic for proofreading my paper. Their comments have dramatically improved the content and the style of this thesis. Henry Cohn, furthermore,voluntarily helped me draw the stratification diagrams in this paper.

I would like to express my gratitude towards my friends who have been very supportive of my work. They include Peter, Richard, Jorge, Jung, Bom, Jihwan, and Taeho among many others. They have been extremely patient in dealing with me this entire year.

My loving parents have provided me with funding to attend Harvard University without which much of my mathematical learning would not have been possible. In addition, I thank the Harvard College Research Program and Dean's Summer Award Program whose generous grants allowed me to stay at Harvard during the summer of 1999 to study mathematics.

Lastly, I would like to thank my good friend and mentor Joshua Charles Greene '97. His senior thesis "The Existence of Hilbert Schemes" provided the inspiration for the work you see here today. I dedicate this thesis to him.

Contents

Chapter 1

Introduction

The target of this thesis is to establish a foundation for understanding the geometry of Hilbert schemes. As an elementary introduction to studying the Hilbert scheme, this paper is mainly a compilation of the materials the author himself learned through in order to study the subject matter. As such, we shall assume the existence of the Hilbert scheme[1] and will be extensively looking at several specific examples; generalizations will be made in appropriate cases.

Although the existence of Hilbert schemes was first proved in the 1960s by Alexandre Grothendieck, surprisingly little is known about their individual structures. For instance, the structure of the Hilbert scheme compactification of the space of twisted cubics was discovered only in the mid-1980s [28]. Even the Hilbert schemes parametrizing zero-dimensional subschemes of $\mathbf{P}^n$ are not clearly understood except in rudimentary cases.

Much of the literature today on this topic focuses on the Hilbert schemes of surfaces or points, or of high degree curves. Most books in this area focus on the general Brill-Noether theory instead of looking at more specific examples of curves. The goal of this paper is to summarize what little is known about the Hilbert schemes for low degree curves and present the materials as much as possible in an accessible manner to the reader. While the content of this paper is modeled after the framework used in *Curves in Projective Spaces* by Joseph Harris and David Eisenbud [13], the purpose is to aim at a more introductory audience; consequently, we will be including much more details and justifications. For background, we will only be requiring basic commutative algebra and scheme theory.

Comments on Notation. The reader should make a note of the following conventions which shall be used throughout this paper.

1. $\mathbf{P}^3$ denotes $\mathbf{P}^3_k$ where k is an algebraically closed field.
2. Projectivization of a vector bundle $\mathbf{P}(V)$ will be a pre-Grothendieckian notation: if E is a vector bundle, then $\mathbf{P}(E)$ is the space of one-dimensional subspaces of the fibers of E, and is equal to $\text{Proj}(\oplus\text{Sym}^n E^*)$.
3. If $\mathcal{F}$ is a coherent sheaf on a scheme X, we will use $h^i(\mathcal{F})$ to denote $\dim H^i(\mathcal{F})$.
4. $\mathcal{H}_{d,g,r}$ denotes the Hilbert scheme parametrizing subschemes of degree d, genus g in $\mathbf{P}^r$. It is often denoted as Hilb^{dn-g+1}.
5. Since we are only looking at the curves in $\mathbf{P}^3$, all curves are embedded in $\mathbf{P}^3$ in this paper; unless specified otherwise $\mathcal{O}_C(1) = \mathcal{O}_{\mathbf{P}^3}(1)|_C$.
6. Unless specified otherwise, we will be using x, y, z for affine coordinates and X, Y, Z, W for projective coordinates.
7. Unless specified otherwise, all genuses will be arithmetic genuses, corresponding to $1 - P_C(0)$ where $P_C(m)$ is the Hilbert polynomial of $C \subset \mathbf{P}^3$.

[1] In this paper, we will merely recall the algebraic properties of the Hilbert scheme (specifically in Section 1.3). Standard references are [23], [10], or [32]. Joshua Greene's senior thesis [9], which is a more digestible presentation of [23], is another excellent reference. What we are interested in is rather how we could use these properties to find out exactly what these Hilbert schemes look like.

8. $\mathcal{G} := G(1,3)$ is the Grassmannian of lines in $\mathbf{P}^3$. It also equals Grass(2,4).

1.1 Motivation

As the title suggests we will be studying the Hilbert schemes parametrizing curves in $\mathbf{P}^3$. More specifically we will be looking at the curves of degrees 1,2, and 3, and of varying genuses. Before we formally define what a Hilbert scheme is, we rightly seek some motivation behind this often-complicated algebraic geometric structure.

One way to view the Hilbert schemes is, as Strømme points out in [32], as a generalization of the Grassmannian varieties. Recall that in projective spaces, linear k-dimensional subspaces of $\mathbf{P}^r$ are parametrized by the Grassmannian variety $G(k,r)$. For instance, the set of lines in $\mathbf{P}^3$ correspond one-to-one with $G(1,3)$. Suppose we were to generalize this notion of parametrization to *subschemes* of $\mathbf{P}^r$, not just subspaces. In other words, we want to be able to study a variety whose points parametrize the subschemes of $\mathbf{P}^r$.

Of course, we cannot even get started on the notion of a parameter space unless we have a meaningful method of specifying the set of subschemes we want to parametrize. The subschemes of our interest could be all algebraic *curves*, for instance. Let's take that example. If we were to study all algebraic curves of $\mathbf{P}^r$, what are the things we would be needing a priori? At the very least, we would need a systematic method of specifying not only the *set* of curves but also the *relations* between two different curves. Already we see the need for a *family* of algebraic curves. So a natural way is to have the family of curves lying in $\mathbf{P}^r \times T$ where T is some variety and for each point of t we assign a curve. To rephrase, we define an *algebraic family* as a subscheme $\mathcal{Z} \subset \mathbf{P}^r \times T$ such that each fiber $\mathcal{Z}_t$ for $t \in T$ is a curve lying in $\mathbf{P}^r \times \{t\} \cong \mathbf{P}^r$. And when this happens, the variety T is called a *parameter space* for our family.

If we are interested in classification problems, we need more than to examine a single family of curves. Even with the Grassmannian varieties, Grass(r,V) is a universal scheme for r-dimensional subspaces of a vector space V: given a k-scheme S, there is a naturally defined one-to-one correspondence between the set of vector bundles $E \longrightarrow S$ that are vector subbundles of the direct product $S \times_k V$ and the set of morphisms from S to Grass(r,V). To continue the notion of generalizing the Grassmannian varieties, we would naïvely hope that our so-called Hilbert scheme would be a universal scheme of a given family of curves or other subschemes. The ideal scenario would be where there is some kind of set of curves $\mathcal{H}$ (of given degree and genus) such that for any other family ($\mathcal{Z} \subset \mathbf{P}^r \times T$) of curves of the same degree and genus we have a map

$$\phi_{\mathcal{Z}} : T \longrightarrow \mathcal{H}$$

such that $\phi_{\mathcal{Z}}(t) = [\mathcal{Z}_t]$ where $[\mathcal{Z}_t] \in \mathcal{H}$ corresponds to the point parametrizing the curve $\mathcal{Z}_t$. In addition, we would want $\phi_{\mathcal{Z}}$ to be a morphism of varieties. In this case, the notion of *universal family* from the Grassmannian varieties would correspond to a closed subscheme $W \subset \mathbf{P}^3 \times \mathcal{H}$ such that for each $h \in \mathcal{H}$, the fiber W_h corresponds to the curve (or subscheme) parametrized by the point h. Grothendieck's construction in [10] basically tells us that all this is possible modulo some conditions.

What, then, are some reasonable conditions? In order to have the ease of manipulating the algebraic properties of the curves, we should do well to consider all this in the category of schemes instead of in the category of varieties. In this thesis, a *curve* will mean any 1-dimensional subscheme embedded in $\mathbf{P}^3$. Focusing our attention only to nonsingular curves is dangerous because singularities arise too easily in practice as limits of nonsingular curves, and our universal family, then, would not be a closed subscheme. For instance, if we consider the family of curves which are pairs of two skew lines, the family of singular conics is bound to be a closed subset; and restricting our attention to smooth curves would mean we would have to take the open compliment of this set.

Furthermore, given a connected component of T, we want our fibers to behave "nicely" without any surprises; one such property is a continuity of the dimension of the fiber. For example, this requirement is violated when we have a projection of the xy-axis mapping onto the x-axis. At the origin, the fiber is 1-dimensional while all other fibers are single points (hence 0-dimensional). This burden is taken care of by insisting that morphisms to the Hilbert scheme correspond to *flat* families. Of course, the price we have to pay is that our notion of limit in the sense of varieties no longer hold; instead, the notion of limit in the sense of the Hilbert scheme is a flat specialization. We will discuss this in detail in Section 2.2 and 2.3. With these conditions in mind, we are just about in a position to understand the Hilbert schemes.

The rest of this chapter will deal with definitions and some well-known properties of the Hilbert schemes. Chapter 2 is mostly background tools and the examples we will be employing later on for studying the Hilbert schemes of curves. In Chapter 3, we look at numerous cases of the Hilbert schemes. Finally Chapter 4 will present the strata of subschemes we study in Chapter 3. Appendix contains some of the computer calculations verifying the saturation of the ideals in Chapter 2 and 4 as well as the flatness of a certain morphism we encounter in Chapter 3.

1.2 Definition of the Hilbert Functor and Hilbert Scheme

Let S be a locally noetherian scheme over k, X a locally projective S-scheme. We define the *functor of points* $h_{X/S}$ of X as the functor that relates the category of schemes to the category of sets in the following manner: given a scheme T, $h_{X/S}(T)$ is the set of S-morphisms $t : T \longrightarrow X$, which we denote as $\mathcal{M}(T, X)$.

Now suppose we are given a functor h from the category of S-schemes to the category of sets. Suppose that we are interested in establishing some kind of equivalence of functors for some S-scheme X. What should we look for? First of all, any morphism of functors $\psi : h_{X/S} \longrightarrow h$ gives rise to a canonical element $\mathcal{X} = \psi(1_X) \in h(X)$. Conversely for an S-scheme X and $\mathcal{X} \in h(X)$, there is an induced morphism $\psi_{\mathcal{X}} : h_{X/S} \longrightarrow h$ such that for any $t : T \longrightarrow X$ in $h_{X/S}(T)$, $\psi := t^*\mathcal{X} \in h(T)$. These constructions give a 1-1 correspondence between $h(X)$ and $\mathcal{M}(h_{X/S}, h)$. When the induced morphism $\psi_{\mathcal{X}}$ is an isomorphism, we say the pair $(X, \mathcal{X})$ *represents* h.[2] Also in this case, h is called a *representable functor* and $\mathcal{X}$ is called the *universal family*. Of course, our construction does not guarantee that such a universal family exists every time.

Let $P \in \mathbf{Q}[t]$ be a polynomial. The *Hilbert functor* is a functor of points such that for a scheme Z, $\mathrm{Hilb}^P(X/S)(Z)$ is the set of closed subschemes $Y \hookrightarrow Z \times_S X$ which are flat over Z, such that their fibers over points of Z have Hilbert polynomial P. (Recall that given a projective subscheme $C \subset \mathbf{P}^n$, its *Hilbert polynomial* is defined as the polynomial $P_C(T) \in \mathbf{Q}[T]$ such that for all sufficiently large r, $P_C(r)$ equals the number of forms of degree r in the homogeneous coordinates of $\mathbf{P}^n$ that are linearly independent over C.) In this case, we want each closed flat family Y to correspond to a morphism from Z to the Hilbert scheme. The *Hilbert scheme* is simply an S-scheme $\mathrm{Hilb}^P(X/S)$ which represents this functor.

It should be remarked that there is an alternative way of defining the Hilbert functor; namely, we can consider a similar functor without the association of a polynomial. However, if Z is a flat family of closed subschemes of X/S parametrized by a connected scheme T, then for all geometric points $t \in$T, the Hilbert polynomials P_{C_t} are the same. This means the Hilbert scheme would necessarily decompose into disjoint union of subschemes indexed by Hilbert polynomials. In other words, the representability of these two functors is equivalent [23]. For the rest of the paper, a Hilbert scheme will always have an associated Hilbert polynomial.

1.3 Basic Properties of the Hilbert Schemes

This section includes basic properties of the Hilbert schemes. Although the contents of these properties are essential to the latter part of this paper, reconstructing these proofs will take us too far away from our focus. For instance, the construction of Hilbert schemes is a nontrivial task. We are content with just citing appropriate references.

Theorem 1.3.1 (The Existence of Hilbert Schemes). *There exists a universal scheme $\mathcal{H}_P$ for the functor $\mathrm{Hilb}^P(X/S)(-)$. It is a locally noetherian, separated, and locally finite type S-scheme. In short, the functor of points of $\mathcal{H}_P$ is the functor $\mathrm{Hilb}^P(X/S)(-)$.*

[2] This definition of a representable functor comes directly from [32].

Reference. [32, Theorem 8.1], [23], or [10]. □

A noteworthy property that comes as a perquisite in the construction is that the Hilbert schemes are both proper and projective [9]. The representability of the functor is indeed a powerful tool; in [17], Robin Hartshorne defines the notion of connected functor and a *fan* (a union of linear subspaces) and uses the representability result to prove that every Hilbert scheme is connected by showing that if the ideals of a fan do not have maximal Hilbert functions, then there is a better choice for a fan; a key element in his proof was Macaulay's Theorem [14, p.18].

The most important property of the Hilbert scheme that we will be using, however, is that we can associate any flat family of curves of given degree d and genus g to a morphism from the parameter space to the Hilbert scheme. This is the universal property of the Hilbert scheme. Firstly, there exists a subscheme $Z \subset \mathcal{H}_{d,g,r} \times \mathbf{P}^r$ whose fiber over $[C]$ (i.e. the point on $\mathcal{H}_{d,g,r}$ parametrizing the subscheme $C \subset \mathbf{P}^r$) is precisely $C \subset \mathbf{P}^r$. Secondly, for any scheme B, the map of sets

$$\mathcal{M}(B, \mathcal{H}_{d,g,r}) \longrightarrow \left\{ \begin{array}{c} \text{subschemes } \mathcal{C} \subset B \times \mathbf{P}^r \text{ whose} \\ \text{fibers over } S \text{ are curves of} \\ \text{degree } d \text{ and genus } g \end{array} \right\}$$

given by $f \longrightarrow (f \times \mathrm{id})^{-1}(Z)$ is a bijection.

This property is the key to understanding how exactly the Hilbert schemes parametrize the flat family of closed subschemes. What follows is an alternative description of $\mathcal{H}_P$. This description also arises from the actual construction of Hilbert schemes.

We know a priori that the Hilbert schemes parametrize the subschemes with a given Hilbert polynomial. Well, then in our case what do these Hilbert polynomials $P_X(m)$ look like? By a well-known theorem [5, III.3], the leading term of the Hilbert polynomial of a scheme X is of the form

$$\frac{\delta(X)}{n!}m^n,$$

where $\delta(X)$ and n are the degree and the dimension of X, respectively. Since we are only concerned with curves, the dimension of X will be 1. Denoting $d := \delta(X)$ as the degree of X, our leading term will be dm.

We also know from the cohomology of coherent sheaves that for a nonsingular curve X

$$\dim_k H^1(\mathcal{O}_X) = g = 1 - P_X(0).$$

In particularly, our Hilbert polynomials, for the most part, look like $P_X(m) = dm - g + 1$. This tells us that restricting our attention to a fixed Hilbert polynomial is consistent with looking at the curves with a fixed genus and degree. At this point, we apply the following lemma.

Lemma 1.3.2 (Uniform m Lemma). *Let $S = k[x_0, ..., x_r]$ and let $O_r(m)$ denote the Hilbert polynomial of $\mathbf{P}^r$ itself. Then $O_r(m) = \binom{r+m}{m} = \dim(S_m)$. For every P, there is an m_0 such that if $m \geq m_0$ and X is a subscheme of $\mathbf{P}^r$ with Hilbert polynomial P_X, then:*

1) $I(X)_m$ is generated by global sections and $I(X)_{l \geq m}$ is generated by $I(X)_m$ as an S-module.

2) $h^i(X, I_X(m)) = h^i(X, \mathcal{O}_X(m)) = 0$ for all $i > 0$.

3) $\dim(I(X)_m) = \binom{r+m}{m} - P_X(m)$, $h^0(X, \mathcal{O}_X(m)) = P_X(m)$ and the restriction map $r_{X,m} : S_m \to H^0(X, \mathcal{O}_X(m))$ is surjective.

Reference. [14, Lemma 1.11]. □

This lemma, combined with the fact the Hilbert function and Hilbert polynomial coincide after some m, tells us two things: firstly, the subspace

$$\Lambda_C = H^0(\mathbf{P}^r, I_C(m)) \subset H^0(\mathbf{P}^r, \mathcal{O}(m))$$

is the space of polynomials of degree m vanishing on C, and has codimension equal to $dm - g + 1$ (for a large m); secondly, by Uniform m Lemma, the subscheme $C \subset \mathbf{P}^r$ is determined by Λ_C.

We take this idea one step further to associate Λ_C as a point in the Grassmannian $G := G(\binom{m+r}{r} - (md - g + 1), \binom{m+r}{r})$. The locus of points in G arising in this way is precisely the scheme $\mathcal{H}_P$. Hilbert schemes parametrize the curves in terms of the hyperplanes that contain the curve. It is not immediately clear why such a locus should come equipped with a scheme structure; and indeed, verifying this is what requires most of the efforts in Mumford's construction.

A sensible reader might have noticed some problems that might arise from this construction. Namely, there are many different ways a curve (or a scheme) could have $P_X(m) = dm - g + 1$ as the Hilbert polynomial. Take a curve C of degree d and genus g, for example. $P_C(m)$ is definitely $dm - g + 1$. Now suppose we have another curve C' which has genus $g + 1$, and we consider $C' \cup \{p\}$ where p is any point in $\mathbf{P}^r$ not lying on the support of C'. This is a subscheme of $\mathbf{P}^r$ having $P_{C' \cup \{p\}}(m) = dm - (g + 1) + 1 + 1 = dm - g + 1$.

What's the catch? Well, there really isn't one. The fact of the matter is that each Hilbert scheme $\mathcal{H}_P$ corresponding to $P(m) = dm - g + 1$ will contain components whose general point corresponds to a curve of genus $g' > g$ and a collection of $g' - g$ points of $\mathbf{P}^r$, if such a configuration is viable. We shall call these components the *ghost components*. This is the extrinsic pathology of the Hilbert schemes that makes them so difficult to study. In most cases, the Hilbert scheme of our interest will contain extra components we are not interested in; what's even more is, these components usually have dimensions greater than that of the actual component of our interest. Harris and Morrison [14] phrase this as the Murphy's Law of Hilbert schemes:

Law 1.3.3 (Murphy's Law for Hilbert Schemes). *There is no geometric possibility so horrible that it cannot be found generically on some component of some Hilbert scheme.*

For this reason, we have to be very careful about these components in our further study of $\mathcal{H}_P$. In fact, Hilbert schemes turn out to be almost never irreducible or smooth. Mumford came up with a famous example [14, 1. D] of the Hilbert scheme parametrizing degree 14 curves with genus 24 in $\mathbf{P}^3$. Such schemes have the Hilbert polynomial $14m - 23$, but there are several components to $\mathcal{H}_{14,24,3}$. While the dimension of the Hilbert scheme is 56, the tangent space has dimension 57 even at a generic point. In other words $\mathcal{H}_{14,24,3}$ is everywhere nonreduced. The examples we will be looking into are admittedly simpler, but we will also run into some complications like this, usually at the intersection of different components of the Hilbert scheme.

Chapter 2

Background Tools

This chapter contains the background tools and basic machinery we will be requiring in order to understand the geometry of $\mathcal{H}_{d,g,r}$. Section 2.1 presents a number of normal bundle calculations that will ultimately help us predict the dimensions of the Hilbert schemes. Section 2.2 is a set of examples of nonreduced schemes that occur as flat limits of smooth curves. Later in Chapter 4, when we derive the complete strata of curves in $\mathbf{P}^3$ (of degrees 2 and 3) in more detail, all of these examples will come in handy. Section 2.3 looks at two particular types of nonreduced schemes (namely the double lines and triple lines) closely. Section 2.4 is a collection of major theorems we will be employing a number of times in Chapter 3. Here again, since these are standard theorems that appear in algebraic geometry ubiquitously, proofs will be omitted but appropriate references will be given.

2.1 Normal Bundles

Definition 2.1.1. Let X be a nonsingular variety over k, and let $Y \subset X$ be an irreducible subscheme defined by a sheaf of ideals $\mathcal{I}$. We define the *normal sheaf* of Y in X, denoted $\mathcal{N}_{Y/X}$ as the dual to the *conormal sheaf* $\mathcal{I}/\mathcal{I}^2$. In other words,

$$\mathcal{N}_{Y/X} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}, \mathcal{O}_Y).$$

If Y is nonsingular, then both the conormal sheaf and the normal sheaf are locally free. In this case $\mathcal{N}_{Y/X}$ is called the *normal bundle.*

Remark 2.1.2. We will encounter cases where $\mathcal{N}_{Y/X}$ is not locally free, but we will still refer to them as normal bundles nonetheless. This abuse of notation should not confused any reader.

Example 2.1.3. *Hypersurfaces of degree n in $\mathbf{P}^3$.* Suppose S is a hypersurface of degree n in $\mathbf{P}^3$ with the ideal sheaf $\mathcal{I}_S$. Then

$$\begin{aligned} \mathcal{I}_S/{\mathcal{I}_S}^2 &= \mathcal{I}_S \otimes \mathcal{O}_{\mathbf{P}^3}/\mathcal{I}_S = \mathcal{I}_S \otimes \mathcal{O}_S = \mathcal{O}_{\mathbf{P}^3}(-n) \otimes \mathcal{O}_S \\ &= \mathcal{O}_{\mathbf{P}^3}(-n)|_S = \mathcal{O}_S(-n). \end{aligned}$$

From this we have that $\mathcal{N}_{S/\mathbf{P}^3} = \mathcal{O}_S(n)$. In particular, we have: 1) $\mathcal{N}_{\mathbf{P}^n/\mathbf{P}^{n+1}} = \mathcal{O}_{\mathbf{P}^n}(1)$, and 2) if Q is a quadric surface, $\mathcal{N}_{Q/\mathbf{P}^3} = \mathcal{O}_Q(2)$.

Now assume $Y \subset X$ is a nonsingular variety. We have the following exact sequence:

$$0 \to \mathcal{I}_Y/\mathcal{I}_Y^2 \to \Omega_{X/k} \otimes \mathcal{O}_Y \to \Omega_{Y/k} \to 0.$$

Taking the dual of this sequence, we get another exact sequence:

$$0 \to \mathcal{T}_Y \to \mathcal{T}_X \otimes \mathcal{O}_Y \to \mathcal{N}_{Y/X} \to 0.$$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}_C & \xrightarrow{\text{id}} & \mathcal{T}_C & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}_S|_C & \longrightarrow & \mathcal{T}_{\mathbf{P}^3}|_C & \longrightarrow & \mathcal{N}_{S/\mathbf{P}^3}|_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \text{id} & & \\
0 & \longrightarrow & \mathcal{N}_{C/S} & \longrightarrow & \mathcal{N}_{C/\mathbf{P}^3} & \longrightarrow & \mathcal{N}_{S/\mathbf{P}^3}|_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Figure 2.1: The Exact Sequence of Normal Bundles

But this is just another way of writing

$$0 \to \mathcal{T}_Y \to \mathcal{T}_X|_Y \to \mathcal{N}_{Y/X} \to 0, \tag{2.1}$$

where "$|_Y$" represents the restriction to Y. Now we suppose that a curve C lies on some surface S which in turn, is contained in $\mathbf{P}^3$. Then we can combine these two exact sequences to get Figure 2.1. The bottom horizontal sequence is of our primary interest since it gives an exact sequence containing $\mathcal{N}_{C/\mathbf{P}^3}$, which will ultimately give us the tangent space to the Hilbert schemes.

Grothendieck's theorem (Theorem 2.4.5) tells us that any vector bundle on $\mathbf{P}^1$ splits. In particular, $\mathcal{N}_{C/\mathbf{P}^3}$ is a direct sum of two line bundles, i.e. it is of the form $\mathcal{O}_{\mathbf{P}^1}(a) \oplus \mathcal{O}_{\mathbf{P}^1}(b)$ for some integers a and b. We recall the following theorem which is first proved by Harris and Hulek using the infinitesimal variation of Hodge structure technique.

Theorem 2.1.4 (Harris & Hulek). *Let C be a smooth curve on some smooth surface $S \subset \mathbf{P}^3$. Then the exact sequence*

$$0 \longrightarrow \mathcal{N}_{C/S} \longrightarrow \mathcal{N}_{C/\mathbf{P}^3} \longrightarrow \mathcal{N}_{S/\mathbf{P}^3}|_C \longrightarrow 0$$

splits if and only if C is a complete intersection of S with some other surface S'.

Reference. [15, p. 129]. □

This is an extremely useful fact since any smooth plane curve can be realized as a complete intersection of the plane and a smooth surface in $\mathbf{P}^3$. We look at some examples to see how to calculate these normal bundles in practice.

Example 2.1.5. *Lines in* $\mathbf{P}^3$. Let's take a line $L \in \mathbf{P}^3$. Since L is a complete intersection of two hyperplanes, we have

$$\begin{aligned}\mathcal{N}_{L/\mathbf{P}^3} &= \mathcal{N}_{L/\mathbf{P}^2} \oplus \mathcal{N}_{\mathbf{P}^2/\mathbf{P}^3}|_L = \mathcal{N}_{\mathbf{P}^1/\mathbf{P}^2} \oplus \mathcal{N}_{\mathbf{P}^2/\mathbf{P}^3}|_L \\ &= \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^2}(1)|_{\mathbf{P}^1} = \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1).\end{aligned}$$

This calculation just confirms that the normal bundle of a line can be obtained from the restriction of the normal bundles of two 1-planes, each of which equals $\mathcal{O}_{\mathbf{P}^2}(1)$.

Example 2.1.6. *Conics.* We can do a similar calculation for a smooth conic C, which is complete intersection of a smooth quadric and a plane in $\mathbf{P}^3$.

$$\mathcal{N}_{C/\mathbf{P}^3} = \mathcal{N}_{C/\mathbf{P}^2} \oplus \mathcal{N}_{\mathbf{P}^2/\mathbf{P}^3}|_C = \mathcal{O}_{\mathbf{P}^2}(2)|_C \oplus \mathcal{O}_{\mathbf{P}^2}(1)|_C = \mathcal{O}_{\mathbf{P}^1}(4) \oplus \mathcal{O}_{\mathbf{P}^1}(2).$$

The last equality comes as a result of the Segre embedding of $\mathbf{P}^1 \hookrightarrow \mathbf{P}^2$.

Example 2.1.7. *Plane Cubics.* Analogously, we can also calculate the normal bundles for a smooth plane cubic $C \in \mathbf{P}^3$. The reader can check that

$$\mathcal{N}_{C/\mathbf{P}^3} = \mathcal{O}_C(3) \oplus \mathcal{O}_C(1).$$

More generally, given a complete intersection of surfaces of degree n and m (and assume $n \leq m$), of equations $f = 0$ and $g = 0$ respectively, the normal bundle $\mathcal{N}_{C/\mathbf{P}^3} = \mathcal{O}_C(n) \oplus \mathcal{O}_C(m)$ [2, 10.2]. How can we see this? Consider the following resolution of $\mathcal{I}_C$:

$$0 \longrightarrow \mathcal{O}_C(-n-m) \longrightarrow \mathcal{O}_C(-n) \oplus \mathcal{O}_C(-m) \longrightarrow \mathcal{I}_C \longrightarrow 0.$$

The first nontrivial map is given by sending a generator α of $\mathcal{O}_C(-n-m)$ to $(\alpha g, -\alpha f)$, and the second map is given by sending (λ, μ) to $\lambda f + \mu g$. If we apply the $\mathrm{Hom}_{\mathcal{O}_C}(\ \ , \mathcal{O}_C)$ functor, the transpose of the first map vanishes since both f and g vanish on C, and we get

$$0 \longrightarrow \mathcal{N}_{C/\mathbf{P}^3} \xrightarrow{\cong} \mathcal{O}_C(n) \oplus \mathcal{O}_C(m) \longrightarrow 0.$$

Example 2.1.8. *Normal Bundle of the Diagonal.* Let us now take a look at $\mathcal{N}_{\triangle/X\times X}$, the normal bundle of the diagonal $\triangle \subset X \times X$. Noting that $\mathcal{T}_{X\times X} \cong \mathcal{T}_X \oplus \mathcal{T}_X$, we have the following exact sequence

$$0 \to \mathcal{T}_X \to \mathcal{T}_X \oplus \mathcal{T}_X \to \mathcal{T}_X \to 0$$

whose two nontrivial maps are given by $\alpha(v) = v \oplus v$ and $\beta(v, w) = v - w$. But $\mathcal{T}_X \cong \mathcal{T}_\triangle$. So in light of the usual exact sequence from (2.1), we conclude that $\mathcal{N}_{\triangle/X\times X} = \mathcal{T}_X$.

Example 2.1.9. *Twisted Cubic Curves.* Calculating normal bundles for curves which are not complete intersections can become difficult. For instance, twisted cubic curves lie on quadric surfaces but are only locally complete intersections. However, in this particular situation, we can still calculate it without too much trouble. Let C be a twisted cubic curve lying on Q a quadric surface in $\mathbf{P}^3$. Then we have

$$0 \longrightarrow \mathcal{N}_{C/Q} \longrightarrow \mathcal{N}_{C/\mathbf{P}^3} \longrightarrow \mathcal{N}_{Q/\mathbf{P}^3}|_C \longrightarrow 0$$

as usual. Suppose furthermore that C is of type $(2,1)$. Then $\mathcal{N}_{C/Q} = \mathcal{O}_C(4)$. This can be seen by looking at the self-intersection of curves of type $(2,1)$. (For instance, one could draw two vertical lines and one horizontal line to represent this, and move the whole configuration around. There are four points of intersection in general.) Since the surface is quadric,

$$\mathcal{N}_{Q/\mathbf{P}^3}|_C = (\mathcal{O}_{\mathbf{P}^3}(2)|_Q)|_C = \mathcal{O}_{\mathbf{P}^3}(2)|_C.$$

Since $\deg(C) = 3$, we have $\mathcal{N}_{Q/\mathbf{P}^3}|_C = \mathcal{O}_C(6)$. (And this is indeed one instance where we violate the notational convention of $\mathcal{O}_C(n)$.) So far we have

$$0 \longrightarrow \mathcal{O}_C(4) \longrightarrow \mathcal{N}_{C/\mathbf{P}^3} \longrightarrow \mathcal{O}_C(6) \longrightarrow 0.$$

Now we know $\mathcal{N}_{C/\mathbf{P}^3} = \mathcal{O}_C(a) \oplus \mathcal{O}_C(b)$ and from the above sequence $a + b = 10$. This can be checked by looking at the degrees or by taking the cohomology. Without loss of generality, assume $a \leq 5$.

Lemma 2.1.10. *$a \geq 4$.*

Proof. Recall that given two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$, $\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) = H^0(\mathcal{L}_1^\vee \otimes \mathcal{L}_2)$ [16, Proposition III.6.7]. In particular, given a 1-dimensional subscheme C,

$$\mathrm{Hom}(\mathcal{O}_C(r), \mathcal{O}_C(s)) = H^0(\mathcal{O}_C(s-r)) = \begin{cases} 0 & \text{if } s < r \\ k^{s-r+1} & \text{if } s \geq r \end{cases}$$

So if $a < 4$, then $O_C(4)$ would have to map into $O_C(b)$, but then $O_C(6)$ is certainly not the cokernel of this map, which should consist of skyscraper sheaves. Another way to get a contradiction is that the projection map $\mathcal{O}_C(a) \oplus \mathcal{O}_C(b) \longrightarrow \mathcal{O}_C(a)$ must be annihilated if $\mathcal{O}_C(4)$ were to map to $\mathcal{O}_C(b)$. In this case, we have a nontrivial induced map from $\mathcal{O}_C(6) \longrightarrow \mathcal{O}_C(a)$, which is a contradiction since $a \leq b < 6$. So we conclude that $(a, b) \in \{(4,6), (5,5)\}$. ☐

Suppose $a = 4$. But then the sequence would be splitting, contradicting Theorem 2.1.4 since a twisted cubic is *not* a complete intersection. We conclude $\mathcal{N}_{C/\mathbf{P}^3} = \mathcal{O}_C(5) \oplus \mathcal{O}_C(5)$. When the normal bundle is of the form $\mathcal{O}_C(b) \oplus \mathcal{O}_C(b)$, we say the bundle is *balanced*. It turns out that this result of balanced normal bundles holds true in general for high degree curves lying on a smooth quadric surface. Eisenbud and Van de Ven noticed the following result in their paper [6].

Theorem 2.1.11. *A smooth rational space-curve C of degree $n \geq 3$ which is contained in a smooth quadric has normal bundle*

$$\mathcal{N}_C \cong \mathcal{O}_C(2n-1) \oplus \mathcal{O}_C(2n-1).$$

And the normal bundle of a twisted cubic is just the special case when $n = 3$.

Example 2.1.12. *A line with an embedded point.* We could also discuss calculating the normal bundles for curves which are not even locally complete intersections. A line with an embedded point is one such case. In this case, we have to go back to the very definition of the normal bundle to calculate this. We will just exhibit how to calculate the dimension of the space of global sections of the normal bundle. Call this scheme L'. Then $L'(\subset \mathbf{P}^3)$ is contained in a plane, so we have a non-splitting exact sequence:

$$0 \longrightarrow \mathcal{N}_{L'/\mathbf{P}^2} \longrightarrow \mathcal{N}_{L'/\mathbf{P}^3} \longrightarrow \mathcal{N}_{\mathbf{P}^2/\mathbf{P}^3} \longrightarrow 0.$$

We first look at $\mathcal{N}_{L'/\mathbf{P}^2}$. After a suitable change of coordinates, if necessary, we can take the ideal defining L' as (X^2, XY). Then we have

$$H^0(L', \mathcal{N}_{L'/\mathbf{P}^2}) = H^0(\mathbf{P}^3, \mathcal{H}om_{\mathcal{O}_{\mathbf{P}^2}}(\mathcal{I}_{L'}/\mathcal{I}_{L'}{}^2, \mathcal{O}_{\mathbf{P}^2}/\mathcal{I}_{L'})) = \mathrm{Hom}_S(I_{L'}/I_{L'}^2, S/I_{L'}),$$

where the right quantity is the space of S-module homomorphisms with $S = k[X, Y, Z]$. This equality comes from [16, Ex. 5.9.a)]. Since $I_{L'}$ is generated in degree 2, we can write this as

$$\phi : \{X^2, XY\} \longrightarrow \{Z^2, ZX, ZY, Y^2\},$$

with the condition that $Y\phi(X^2) = X\phi(XY)$ modulo $I_{L'}$. Then we find that X^2 has to be mapped to aZX and XY has to be mapped to $bZX + cZY + dY^2$. So we conclude $h^0(L', \mathcal{N}_{L'/\mathbf{P}^2}) = 4$. Then

$$h^0(L', \mathcal{N}_{L'/\mathbf{P}^3}) = h^0(L', \mathcal{N}_{L'/\mathbf{P}^3}) + h^0(L', \mathcal{N}_{\mathbf{P}^2/\mathbf{P}^3}) = 4 + 3 = 7.$$

We shall see later that this corresponds to the tangent space of the Hilbert scheme which parametrizes this class of schemes.

Example 2.1.13. *Triple lines.* Our final example is calculating the dimension of the space of global sections of the normal bundle of a triple line whose ideal is given by a square on an ideal defining a line. For instance, if we let T be this scheme, $I_T = (X^2, XY, Y^2)$. Carrying the same line of argument as in Example 2.1.12, we have $H^0(T, \mathcal{N}_{T/\mathbf{P}^3})$ as the space of S-module homomorphism where S now is $k[X, Y, Z, W]$. I_T is generated in degree 2, so again we have

$$\phi : \{X^2, XY, Y^2\} \longrightarrow \{XZ, XW, YZ, YW, Z^2, ZW, W^2\}$$

where $Y\phi(X^2) = X\phi(XY) \mod I_T$ and $Y\phi(XY) = X\phi(Y^2) \mod I_T$. With these conditions imposed, we conclude

$$H^0(T, \mathcal{N}_{T/\mathbf{P}^3}) = \mathrm{Hom}(\{X^2, XY, Y^2\}, \{XZ, XW, YZ, YW\}),$$

which is 12 dimensional.

2.2 Non-Reduced Schemes I: Examples of Embedded Points

The construction of the Hilbert scheme requires that we parametrize *flat* families of subschemes. In this sense, a Hilbert scheme is a compactification – it includes all of the flat limits of given configurations of curves. That is to say, if we have a 1-parameter family C_t of curves, the limit in the sense of Hilbert schemes of C_t as $t \longrightarrow t_0$ is defined by

$$\lim_{t \longrightarrow t_0} I(C_t).$$

Often the resulting curve may not necessarily agree with C_{t_0} other than having the common support. This section will deal with singular flat limits occurring from 1-parameter families of smooth curves. (Recall that 2-parameter families may not exhibit flat limits.) We shall pay particular attention to *embedded points* in this section.

The second objective of this section is to exhibit some explicit calculations in detail, so that in Chapter 4 when we construct the strata of curves, we may take some of the specializations for granted. Since we are only interested in the local geometry it is enough to look at the affine coordinates. In all of these examples we will be studying the 1-parameter limit of a certain ideal intersecting one or more ideals. All of the examples listed here can be done with simple manipulation of generators of the given ideals. But if the example becomes more complicated, in order to convince ourselves that there are no more generators other than what can be obtained from simple manual manipulations and speculations, we need to use the process of *saturation*, otherwise known as Buchberger's Algorithm. For more information on this algorithm, refer to [3, 2.8–2.9] or [4, p. 362]. The computer program MACAULAY2 is an effective means of verifying the process of saturation. Appendix A includes the MACAULAY2 commands (and verifications) for each of these examples.

We begin with Harris and Eisenbud's definition of *embedded components* [5, pp. 66–70].

Definition 2.2.1. A scheme $X = \text{Spec } k[x_1, ..., x_n]/I \subset \mathbf{A}_k^n$ has an *embedded component* if for some open subset $U \subset \mathbf{A}_k^n$ meeting X in a dense subset of X the closure of $X \cap U$ in X does not equal X. This is equivalent to saying that the primary decomposition of the ideal I contains embedded primes. An *embedded point* occurs when the embedded prime is maximal.

Example 2.2.2. Suppose you consider the family of a line and a point in $\mathbf{A}_k^2$. For simplicity, let L be defined by the ideal (x) and the point P_t by $(x-t, y-t)$. We let $X_t = L \cup P_t$, and consider $\lim_{t \longrightarrow 0} X_t$. (In each of the subsequent examples in this section X_t will always denote the union of two schemes at least one of which is parametrized by t.) Since $I(X_t) = I(L) \cdot I(P_t) = (x^2 - xt, xy - xt)$ for $t \neq 0$, we conclude that

$$I(X_0) = (x^2, xy) = (x) \cap (x, y)^2 = I(L \cup P_0) \cap (x, y)^2$$

and we get an embedded point at the origin. Notice that in this case the embedded point does *not* preserve the direction from which P_t was approaching L. We will, however, see in the next example that this is not the case in $\mathbf{A}_k^3$.

Example 2.2.3. Now suppose we have $I(L) = (x, z)$ and $I(P_t) = (x - at, y - bt, z - ct)$ in $\mathbf{A}_k^3$. Then by an analogous argument we immediately have $(x^2, xz, yz, yz, z^2) \subset I(X_0)$. In addition notice that for $t \neq 0$ $I(X_t)$ also contains

$$\frac{1}{t}(x(z - ct) - z(x - at)) = az - cx.$$

So we conclude

$$I(X_0) = (x^2, z^2, zx, zy, xy, az - cx).$$

Here for example, it might not be clear why we are done. The process of *saturation* would entail considering the ideal $(x, z) \cap (x - at, y - bt, c - zt)$ as an ideal in the polynomial ring $k[x, y, z, t, t^{-1}]$, generated by this ideal, intersecting this ideal to get the bigger ideal I in $k[x, y, z, t]$, and then reducing modulo t. We note that the embedded point at the origin preserves the information of the plane (spanned by the point and the tangent line to the curve (in this case, a line) at infinitesimally small t) but *not* the direction within that plane. In general, specifying this type of *spatial* embedded point (i.e. emerging from the plane) will amount to specifying a point on the line and a normal direction. Notice here that given the coordinate system, *this is the unique way of writing down this ideal modulo scaling.* This is because $az - cx$ is the only linear term, and it cannot be represented by any other linear terms. Therefore the space of all embedded points in a given line and a point on it is isomorphic to $\mathbf{P}^1$. This turns out to be an important fact later on.

There are other ways of obtaining embedded points without any moving point P_t approaching a fixed line.

Example 2.2.4. Let L_t be the line in $\mathbf{A}_k^3$ defined by the ideal $(y, z-t)$ and M be the line defined by (x, z); for $t \neq 0$ let X_t be their union. We ask what the flat limit of X_t is as $t \longrightarrow 0$. When $t \neq 0$, we have

$$I(X_t) = I(L_t) \cdot I(M) = (y, z-t) \cdot (x, z) = (yx, yz, zx - zt, z^2 - zt).$$

Taking the limit as $t \longrightarrow 0$ we get

$$X_0 = \operatorname{Spec} k[x, y, z]/(z^2, xz, yz, xy),$$

instead of $I(L_0 \cup M) = \operatorname{Spec} k[x, y, z]/(z, xy)$. Notice that

$$I_{X_0} = (z, xy) \cap (x, y, z)^2 = I(L_0 \cup M) \cap (x, y, z)^2.$$

The limit scheme X_0 has an embedded point at the origin. Here the Zariski tangent space has dimension 3.

Example 2.2.5. Suppose you have three nonplanar concurrent lines. We now study what happens when one of the line rotates towards the plane defined by the other two lines. We are interested in seeing the scheme structure of three coplanar concurrent lines that arise as a specialization of three nonplanar concurrent lines. Let L, M, N_t be defined by the ideal $(y, z), (x, z), (x - y, z - tx)$ respectively. At $t = 0$ the reduced scheme of the set of three lines would lie in the plane $z = 0$. Let X_t denote the union of three of these lines. Since the three coordinate axes are generated by (xy, xz, yz) we have $I(X_t) = (Q_1, Q_2, Q_3)$ where $Q_1 = z(z - tx), Q_2 = z(z - ty), Q_3 = (z - tx)(z - ty)$. When $t \longrightarrow 0$, all of the Q_i's approach z^2, so $z^2 \in I(X_0)$. In addition, notice also that when $t \neq 0$, $I(X_t)$ contains

$$\frac{Q_1 - Q_3}{t} = yz - txy \quad \text{and} \quad \frac{Q_2 - Q_3}{t} = xz - txy,$$

and therefore $I(X_0)$ contains xz and yz. Finally, $I(X_t)$ also contains

$$x\frac{Q_1 - Q_3}{t} - y\frac{Q_2 - Q_3}{t} = txy(x - y).$$

We conclude that $I(X_0)$ also contains $xy(x - y)$. Noting that the dimension of the Zariski tangent space is an upper-semicontinuous function, we can safely conclude that

$$I(X_0) = (xz, yz, z^2, xy(x - y)) = (z, xy(x - y)) \cap (x, y, z)^2 = I(L \cup M \cup N_0) \cap (x, y, z)^2,$$

whence the limit scheme X_0 has an embedded point.

Example 2.2.6. There are also examples of nonreduced schemes which do not have embedded points. We include one such example in this section for future reference. Double and triple lines are good examples – the following is an example of a double line. Suppose you have

$$I(L) = (x, y) \quad \text{and} \quad I(M_t) = (x + \alpha(t)z + \beta(t), y + \gamma(t)z + \delta(t)),$$

such that all of the t functions vanish at $t = 0$ and are differentiable there. Then for $t \neq 0$

$$I(X_t) = I(L) \cdot I(M_t) = \begin{bmatrix} x^2 + \alpha xz + \beta x \\ xy + \gamma xz + \delta x \\ xy + \alpha yz + \beta y \\ y^2 + \gamma yz + \delta y \end{bmatrix}.$$

As usual we take the limiting value of each term as a generator. In addition we have

$$\lim_{t \longrightarrow 0} \frac{1}{t}\{(xy + \gamma xz + \delta x) - (xy + \alpha yz + \beta y)\} = \gamma' xz + \delta' x - \alpha' yz - \beta' y$$

as an additional generator. So $I(X_0) = (x^2, xy, y^2, \gamma' xz + \delta' x - \alpha' yz - \beta' y)$, which tells us that up to scalars, we obtain a double line as specifying a normal direction at each point on the line. This also means that in general a double line obtained as a flat limit of two approaching lines would not lie in a plane but rather on a quadric surface.

Example 2.2.7. We can also specialize a given double line to get an embedded point. Take the simplest nonplanar double line L where the specified normal directions twist just once around L:

$$I(L) = (x^2, xy, y^2, zy - x).$$

This is essentially the z-axis and at each z_0 the normal direction is a vector on the plane $z = z_0$ with slope equaling $1/z_0$. In particular, at the origin, the normal direction is straight up in the y-axis direction (not in the z-axis direction). Consider a specialization where $(x, y, z) \mapsto (x, ty, z)$. For a given t value, the normal direction at every point except for the origin lies on the plane $z = z_0$ and the slope is $(tz_0)^{-1}$. As $t \longrightarrow \infty$, all normal directions except at the origin will have 0 slope, and hence will lie on the xz-plane. While sending $t \longrightarrow \infty$ was the more intuitive way to see this, we could also send $(x, y, z) \mapsto (sx, y, z)$ and let s go to 0. (Since our field is algebraically closed k or $\mathbf{C}$, there is not a well-defined way of saying s goes from 1 to 0. We will just say $s \longrightarrow 0$.) When this happens, the ideal becomes (x^2, xy, y^2, zy). Hence the double line has an embedded point.

So far all of the embedded points (in $\mathbf{P}^3$) we looked at were *spatial* embedded points. This means the Zariski tangent space at the point is 3, and the embedded point does not lie in the plane. We will see an embedded point of a different type in the next example.

Example 2.2.8. Suppose you have a singular conic C defined by $I(C) = (xy, z)$. And let $P_t = (x - t, y - t, az - bt^2)$. Similarly we can ask what $I(X_0)$ is. For $t \neq 0$ we have, as generators,

$$x^2y - txy, xy^2 - txy, axyz - bt^2xy, zx - zt, zy - zt, az^2 - bt^2z.$$

In addition, we also have

$$\frac{1}{t^2}\{(axyz - bt^2xy) - ay(zx - zt) - at(zy - zt)\} = az - bxy.$$

We conclude that

$$I(X_0) = (x^2y, xy^2, zx, zy, z^2, az - bxy).$$

If $a = 0$, then the ideal reduces down to (zx, zy, xy, z^2), which is just a spatial embedded point whose Zariski tangent space at the point is 3-dimensional. If $b = 0$, then the ideal reduces down to (x^2y, xy^2, z) and this is a *planar* embedded point and the ideal consists of functions whose restriction to the plane $z = 0$ vanish with order 3 at the origin. When both a and b are nonzero, we will call this a *general* embedded point. We again note that given a specific coordinate system, and once we specify our embedded point to be written in terms of $az - bxy$, we have a unique way of representing it as $[a, b] \in \mathbf{P}^1$. So the space of all the embedded points (at the node) given a particular singular cubic is also isomorphic to $\mathbf{P}^1$ (See Example 2.2.3).

2.3 Non-Reduced Schemes II: Classification of Double Lines and Triple Lines in $\mathbf{P}^3$

It turns out that all of the Hilbert schemes corresponding to the Hilbert polynomial $P(m) = 2m + c$, (here $c \geq 1$) have a component whose points parametrize double lines. While in affine coordinates all double lines are isomorphic to each other [5, II.3.5], that is not the case in projective spaces. We define a *double line* to be the scheme with multiplicity 2 structure whose support is a line in $\mathbf{P}^r$ (in our case $r = 3$). We saw an example of this kind of line in Example 2.2.6. For the remainder of this section, our double lines and triple lines will have no embedded points. We prove the following proposition.

Proposition 2.3.1. *Let D be the line $\{X = Y = 0\}$ in $\mathbf{P}^3$, and let $a \geq -1$ be an integer. Suppose we have two homogeneous polynomials F and G of degree $a + 1$ in terms of Z and W, which have no common zeros along D. Then F and G define a surjection $u : \mathcal{I}_D \longrightarrow \mathcal{O}_D(a)$ by $X \mapsto F$ and $Y \mapsto G$. The kernel of u gives the ideal sheaf of a multiplicity two structure L on D. We have* *i)* $p_a(L) = -1 - a$

ii) $I_L = (X^2, XY, Y^2, XF - YG)$

iii) If F' *and* G' *define another multiplicity structure* L'*, then* $L = L'$ *if and only if* $F' = cF \bmod I_L$ *and* $G' = cG \bmod I_L$ *for* $c \in k^*$.

iv) Each multiplicity two structure L *on* D *arises by this construction.*

Proof. Since L is of multiplicity two, the ideal must contain $(X, Y)^2 = (X^2, Y^2, XY)$. But this ideal has to be a proper subideal of $I(L)$ since the square of an ideal defining a line is a triple line. So there must be at least another generator $Q(X, Y, Z, W)$. We derive that Q must be of the form written above.

First notice that since we already have X^2, Y^2, and XY contained in $I(L)$, Q can be written in the following form:

$$XF(Z, W) - YG(Z, W) + H(Z, W).$$

But since Q must vanish at $X = Y = 0$ for all Z and W, H must, in fact, be identically 0 (modulo I_L). Now we know that $Q(X, Y, Z, W) = XF - YG$ with F, G as polynomials in Z and W only. Clearly, for Q to be homogeneous, the degrees of F and G need to match. In addition we know F and G have no common zeroes, lest there would be an embedded point.

We want to conclude that $I(L)$ has no more generators other than X^2, Y^2, XY and Q. Suppose we have another generator Q'. Then we know that Q' can also be written as $XF'(Z, W) - YG'(Z, W)$ where F' and G' satisfy similar conditions as F and G.

It now suffices to prove $F/G = F'/G'$. Suppose $F/G \neq F'/G'$. Then we can choose Z_0 and W_0 such that $F(Z_0, W_0)/G(Z_0, W_0) \neq F'(Z_0, W_0)/G'(Z_0, W_0)$. But then at the point $(0, 0, Z_0, W_0) \in L$, Zariski tangent space becomes 1-dimensional since we have two different linear conditions imposed on it. This contradicts the fact that L has the multiplicity 2 structure. So Q is uniquely determined up to multiplication by scalars, and $I(L)$ is of the form stated above. This proves ii), iii) and iv).

To prove i), we can actually determine the genus of any such double line by looking at its Hilbert polynomial. We denote $L = L_\phi$ as the double line structure put on the reduced line $V(X, Y) \in \mathbf{P}^3$ such that

$$\phi([0, 0, Z, W]) = L, [G(Z, W), F(Z, W), 0, 0,],$$

where F and G are homogeneous polynomials of degree $a + 1$ in Z and W. Here the ideal defining L is of the following form:

$$I(L_\phi) = (X^2, XY, Y^2, XF(Z, W) - YG(Z, W)).$$

We calculate the Hilbert polynomial. Given an arbitrary polynomial $N = N(X, Y, Z, W)$ we count the number of conditions imposed on N so that it vanishes on L_ϕ (so that N is contained in $I(L_\phi)$). Suppose the degree of N is $m \gg a + 1$. This will ensure that the Hilbert function we calculate match the Hilbert polynomial. We can write N as

$$N(X, Y, Z, W) = N_{0,0}(Z, W) + X_0 H_{1,0}(Z, W) + X_1 H_{0,1}(Z, W) + \cdots = \sum_{i,j \geq 0} X_0^i X_1^j H_{i,j}(Z, W).$$

Since $I(L_\phi)$ contains X^2, Y^2 and XY, all terms after the first three might as well vanish – they don't actually vanish, but we can ignore them modulo I_L. We first require that N vanish on L. This means $N_{0,0} = 0$, and here we are imposing $m + 1$ conditions. Now we require that N vanishes on the scheme structure defined by L_ϕ. This is satisfied precisely when $XF - YG$ divides $XH_{1,0} + YH_{0,1}$. So we have

$$(XF(Z, W) - YG(Z, W))K(X, Y, Z, W) = XH_{1,0}(Z, W) + YH_{0,1}(Z, W).$$

By analyzing this equation when $X = 0$ and $Y = 0$ separately, we can deduce that K is actually a polynomial (of degree $m - a - 1 - 1$) in Z and W only. Since the space polynomials of two $m - 1$ degree polynomials $(H_{1,0}, H_{0,1})$ is $2m$ but the space of degree $m - a - 1 - 1$ polynomials is $m - a - 1$. In turn, we are imposing $2m - (m - a - 1) = m + a + 1$ more conditions on N. Therefore the Hilbert polynomial is

$$P_{L_\phi}(m) = 2m + a + 2.$$

Since m was chosen to be sufficiently large, this is the actual Hilbert polynomial. In light of this, we conclude the genus of L_ϕ is $1 - P_{L_\phi}(0) = -1 - a$. □

Remark 2.3.2. Notice that implicit in this proposition is that the Hilbert polynomial of a double line L can only be $2m + c$ where $c \geq 1$. So these can only occur in the Hilbert schemes corresponding to these polynomials. In addition, if L has embedded points, the Hilbert polynomial can only increase.

One can do a similar analysis for triple lines. Steve Nollet [25] classified all triple lines in $\mathbf{P}^3$ by looking at the resolutions of the ideal sheaf defining the multiplicity three structure on a line $D \subset \mathbf{P}^3$. He concluded that triple lines can be classified by type (a, b) where $a \geq -1$ and $b \geq 0$. For the case $a \geq 0$, the genus is at most -2. *We shall disregard this case since it is not of our interest.* For $a = -1$ he proves the following proposition, which is a *partial* classification of triple lines in $\mathbf{P}^3$.

Proposition 2.3.3. *Let $D \subset \mathbf{P}^3$ be the line $\{X = Y = 0\}$ and let L be the multiplicity two structure $\{X = Y^2 = 0\}$ on Y. Let P, Q be two homogeneous polynomials of degree $b-1$ and b in (Z, W) which have no common zeros along D. Then P and Q define a surjection $u : \mathcal{I}_L \longrightarrow \mathcal{O}_D(b-2)$ by $X \mapsto P$, $Y^2 \mapsto Q$. The kernel u is sheaf of a multiplicity three structure T on D. Further, we have*

i) $p_a(T) = 1 - b$

ii) $I_T = (X^2, XY, Y^3, XQ - Y^2P)$

iii) If P' and Q' define another three structure T', then $T = T'$ if and only if there exists $c \in k^*$ such that $P' = cP \bmod I_D$ and $Q' = cQ \bmod I_D$.

Remark 2.3.4. For instance, if we are interested in multiplicity 3 structure schemes having genus 1, we know that Q would be have to be a degree 0 polynomial and $P = 0$. So the only multiplicity 3 structure scheme would look like $I_T = (Y^3, X)$. In other words, this is a planar triple line arising as a specialization of a plane cubic.

Remark 2.3.5. We also know that for the genus 0 case, the line would have the ideal of the following form:

$$I_T = (X^2, XY, Y^3, X(cZ + dW) - Y^2e).$$

Here c, d, e are constants, and $e \neq 0$. After a suitable change of coordinates, we can rewrite this as

$$I_T = (X'^2, X'Y', Y'^3, a'Z'X' - Y'^2).$$

If $a' = 0$ then we have $I_T = (X', Y')^2$. Otherwise, we can eliminate Y'^3, and the ideal becomes $I_T = (X'^2, X'Y', a'Z'X' - Y'^2)$, which is a triple line lying on a cone with second order normal varying. We will come back to this in Chapter 4.

2.4 Frequently Used Theorems

The following theorems will be used numerous times in the next chapter.

Theorem 2.4.1 (Zariski's Main Theorem). *Let X be a normal variety over k, and let $f : X' \longrightarrow X$ be a birational morphism with finite fibres from a variety X' to X. Then f is an isomorphism of X' with an open subset $U \subset X$.*

Reference. [24, III.9, p.288]. □

Corollary 2.4.2. *Let $f : X \longrightarrow Y$ be a bijective morphism and suppose X and Y are smooth and* $\dim X = \dim Y$. *Then f is an isomorphism.*

Proof. This follows immediately from Zariski's Main Theorem. □

In Chapter 3, when we use Zariski's Main Theorem, we will mainly be referring to the result stated in Corollary 2.4.2.

Theorem 2.4.3 (Bezout's Theorem). *Let X and $Y \subset \mathbf{P}^n$ be subvarieties of pure dimensions k and l with $k + l \geq n$, and suppose they intersect generically transversely. Then*

$$\deg(X \cap Y) = \deg(X) \cdot \deg(Y)$$

In particular, if $k + l = n$, this says that $X \cap Y$ will consist of $\deg(X) \cdot \deg(Y)$ *points.*

Reference. [12, Theorem 18.3]. □

Theorem 2.4.4 (Riemann-Roch). *Let D be a divisor on a curve X of genus g and degree d. Then*

$$l(D) - l(K - D) = d + 1 - g.$$

Reference. [16, Theorem IV.1.3]. □

Theorem 2.4.5 (Grothendieck). *Every vector bundle E of rank n over $\mathbf{P}^1$ has the form*

$$E = \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_n)$$

with uniquely determined numbers $a_1, ..., a_n \in \mathbf{Z}$ with $a_1 \geq a_2 \geq \cdots \geq a_n$.

Reference. [26, Theorem 2.1.1]. □

Theorem 2.4.6 (Serre Duality Theorem for Curves). *For any line bundle L on a smooth curve C the pairing*

$$\langle , \rangle : H^1(C, L) \times H^0(C, KL^{-1}) \longrightarrow \mathbf{C}$$

given by $\langle \alpha, \beta \rangle = \int_C \alpha \wedge \beta$ is a duality.

Reference. [1, I.2, p. 7]. □

Theorem 2.4.7 (Cohomology and Base Change). *Let $f : X \to Y$ be a projective morphism of noetherian schemes and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. Let $y \in Y$. We have the following results:*

a) If the natural map

$$\phi^i(y) : R^i f_*(\mathcal{F}) \otimes \kappa(y) \to H^i(X_y, \mathcal{F}_y)$$

is surjective, then it is an isomorphism and the same is true for all y' in a neighborhood of y;

b) Assume that $\phi^i(y)$ is surjective. Then, the following are equivalent:

(i) $\phi^{i-1}(y)$ is also surjective;

(ii) $R^i f_*(\mathcal{F})$ is locally free in a neighborhood of y.

c) When the conditions of (b) hold for all $y \in Y$, $R^i f_*(\mathcal{F})$ is of formation compatible with any locally noetherian base change. By this we mean that if $p : Y' \to Y$ is a base change map with Y' locally noetherian, and $f' : X' \to Y$ is a base change map with Y' locally noetherian, and $f' : X' \to Y'$ and $\mathcal{F}'$ are induced by base change, then we have the following isomorphism via the natural map:

$$p^* R^i f_*(\mathcal{F}) \cong R^i f'_*(\mathcal{F}').$$

References. [16, Theorem III.12.11] or [22, II.5]. □

Theorem 2.4.8 (Riemann Extension Theorem). *Suppose $f(z_1, ..., z_{n-1}, w)$ is holomorphic in a disc $\triangle \subset \mathbf{C}^n$ and $g(z_1, ..., z_{n-1}, w)$ is holomorphic in $\overline{\triangle} - \{f = 0\}$ and bounded. Then g extends to a holomorphic function on $\triangle$.*

Reference. [11, p. 9]. □

Corollary 2.4.9. *Suppose $\rho : X \longrightarrow Y$ is a rational map which extends uniquely to a continuous $\rho' : X \longrightarrow Y$. Suppose furthermore that X is smooth. Then ρ' is a regular map.*

Proof. Let D be a subvariety of codimension 1 which contains all the singularities of X. We then have $\rho' : X - D \longrightarrow Y$, which satisfies the conditions required by Riemann Extension Theorem. Since ρ' is continuous it is also bounded. Hence ρ is a holomorphic function (hence regular) in X. □

Theorem 2.4.10 (Universal Property of Blowing Up). *Let $f : Y \to X$ be a morphism of noetherian schemes, and let $\mathcal{I}$ be a coherent sheaf of ideals on X. If w let $\mathcal{I}'$ be the inverse image ideal sheaf $f^{-1}\mathcal{I}$ under f, then there is a unique morphism $f' : Bl_{\mathcal{I}'}Y \to Bl_{\mathcal{I}}X$*

$$\begin{array}{ccc} Bl_{\mathcal{I}'}Y & \overset{f'}{\dashrightarrow} & Bl_{\mathcal{I}}X \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array} \tag{2.2}$$

making a commutative diagram as shown. Moreover, if f is a closed immersion, so is f'.

Reference. [16, Theorem II.7.15]. □

Chapter 3

Looking at Hilbert Schemes

In this chapter we take a close look at the structures of certain Hilbert schemes. In particular we will be exclusively looking at $\mathcal{H}_{d,g,r}$ for $r = 3$ and $d = 1, 2, 3$. As mentioned before, the subschemes of our interests are curves. Unfortunately, if we are interested in the geometry of these Hilbert schemes, there is not always one standard method of attacking these problems. The reader will see that even with the first three structures we study, the approaches are quite different. Sometimes the method amounts to guessing the structure beforehand and exhibiting a morphism from it. As usual, major results will be cited as necessary.

3.1 Dimensions of the Hilbert Schemes

The first question is the upper bound. Here we are relatively lucky in that the functorial property of the Hilbert scheme lets us identify the tangent space $T_{[C]}$ at the point $[C]$ corresponding to the curve $C \in \mathbf{P}^r$. Recall that the tangent space to any scheme S at a closed point p is the set of maps $\mathrm{Spec}(k[\epsilon]/\epsilon^2) \longrightarrow S$ centered at p. Using this we get a description of the tangent space.

Theorem 3.1.1. *For a closed subscheme $C \subset \mathbf{P}^r$, we have*

$$T_{[C]}\mathcal{H} = H^0(C, \mathcal{N}_{C/\mathbf{P}^r}).$$

In other words, the tangent space to the Hilbert scheme $\mathcal{H}$ at the point corresponding to C is isomorphic to the space of global sections of the normal sheaf $\mathcal{N}_{C/\mathbf{P}^r}$.

Reference. [31, Theorem VI.4.4]. □

Now notice that $\dim \mathcal{O}_{\mathcal{H},[C]} \leq \dim T_{[C]}\mathcal{H}$ with equality holding if and only if the ring $\mathcal{O}_{\mathcal{H},[C]}$ is regular (or equivalently, if $\mathcal{H}$ is smooth at $[C]$). This gives an immediate upper bound on the dimension of the Hilbert scheme.

Corollary 3.1.2. $\dim_{[C]} \mathcal{H} \leq h^0(C, \mathcal{N}_{C/\mathbf{P}^r})$.

We turn our attention to establishing a lower bound. Here we use a basic fact about the space of obstructions of C in $\mathbf{P}^r$. The actual theory of obstruction space is rather difficult, and we will not go into it in much detail. [18, p. 21–39] is a standard reference. The main idea behind the obstruction theory is that one can try to give local equations for $\mathcal{H}$. One way to do this is to look at the tangent space at $[C]$ and impose the conditions that an infinitesimal deformation be extended indefinitely. This is to say that that our infinitesimal deformation is not *obstructed.* Then minimum number of these local equations is the dimension of the obstruction space, and we get the following relation of dimensions [18, Theorem I.2.8.]:

$$h^0(C, \mathcal{N}_{C/\mathbf{P}^r}) - \dim o(C, \mathbf{P}^r) \leq \dim_{[C]} \mathcal{H}.$$

Of course, this does not really help us estimate the lower bound on the dimension of the Hilbert scheme unless we know how to estimate the upper bound on the dimension of the obstruction space. However, it turns out that there is an easy way to calculate the upper bound of the dimension of the obstruction space in certain situations.

Proposition 3.1.3. *If C is locally a complete intersection, in particular if it is smooth, then $o(C, \mathbf{P}^r)$ is a subspace of $H^1(C, \mathcal{N}_{C/\mathbf{P}^r})$.*

Reference. [2, p. 449] or [19]. □

Combining all we have, we get the following bound for the dimension of the Hilbert scheme at $[C]$ *where C is locally a complete intersection*:

$$\chi(\mathcal{N}_{C/\mathbf{P}^r}) \leq \dim_{[C]} \mathcal{H} \leq h^0(X, \mathcal{N}_{C/\mathbf{P}^r}),$$

where $\chi(\mathcal{N}_{C/\mathbf{P}^r}) = h^0(C, \mathcal{N}_{C/\mathbf{P}^r}) - h^1(C, \mathcal{N}_{C/\mathbf{P}^r})$. But the left quantity can be computed now. Taking the Euler-Poincare characteristic of the following sequence,

$$0 \to \mathcal{T}_C \to \mathcal{T}_{\mathbf{P}^r}|_C \to \mathcal{N}_{C/\mathbf{P}^r} \to 0,$$

we get $\chi(\mathcal{N}_{C/\mathbf{P}^r}) = \chi(\mathcal{T}_{\mathbf{P}^r}|_C) - \chi(\mathcal{T}_C)$. By Riemann-Roch,

$$\chi(\mathcal{T}_C) = \chi(-K) = \deg(-K) - g + 1 = -(2g-2) - g + 1 = -3(g-1).$$

Meanwhile, we also have the Euler sequence [16, p. 182]:

$$0 \to \mathcal{O}_{\mathbf{P}^r} \to \mathcal{O}_{\mathbf{P}^r}(1)^{r+1} \to \mathcal{T}_{\mathbf{P}^r} \to 0.$$

Restricting this onto C, we have

$$\chi(\mathcal{T}_{\mathbf{P}^r}|_C) = \chi(\mathcal{O}_C(1)^{r+1}) - \chi(\mathcal{O}_C) = (r+1)(d-g+1) - (1-g).$$

The last equality again comes from Riemann-Roch. In sum we finally get

$$\chi(\mathcal{N}_{C/\mathbf{P}^r}) = \chi(\mathcal{T}_{\mathbf{P}^r}|_C) - \chi(\mathcal{T}_C) = (r+1)(d-g+1) - (1-g) - 3(1-g) = (r+1)d + (r-3)(1-g).$$

We call $\chi(d, g, r) := (r+1)d + (r-3)(1-g)$ *the Hilbert number*. In particular, when $r = 3$ the lower bound becomes $4d$. We can do a little better with our conditions, however. Since

$$0 \leq h^1(\mathcal{N}_{C/\mathbf{P}^r}) \leq h^1(\mathcal{T}_{\mathbf{P}^r}|_C) \leq h^1(\mathcal{O}_C(1)^{r+1}) = (r+1)h^1(\mathcal{O}_C(1))$$

if $h^1(\mathcal{O}_C(1)) = 0$ (in other words if the curve is nonspecial) then $h^1(\mathcal{N}_{C/\mathbf{P}^r}) = 0$. We have just proved the following:

Theorem 3.1.4. *If C is a smooth, irreducible, nondegenerate curve of degree d and genus g in $\mathbf{P}^r$ with $\mathcal{O}_C(1)$ nonspecial, then $H^1(C, \mathcal{N}_{C/\mathbf{P}^r}) = 0$ and*

$$\dim_{[C]} \mathcal{H}_{d,g,r} = \dim T_{[C]}\mathcal{H}_{d,g,r} = \chi(d, g, r).$$

In particular, $\mathcal{H}_{d,g,r}$ is smooth in this case.

In addition to this theorem, we note the following lemma.

Lemma 3.1.5. *For curves of genus $0 \leq g \leq 2$, $h^1(\mathcal{O}_C(1)) = 0$.*

Proof. By Serre-Duality on curves, we have $h^1(D) = h^0(K - D)$. Since $\deg K = 2g - 2$, $H^0(K - D)$ vanishes for all curves of genus 0 or 1. For curves of genus 2, notice that all of them still have vanishing $H^0(K - D)$ as long as degree is greater than $2g - 2 = 2$. For curves of degree 1 or 2 in $\mathbf{P}^3$, genus 2 is actually never achieved since the maximum genus for these curves is $\frac{1}{2}(d-1)(d-2) = 0$. Hence $H^1(\mathcal{O}_C(1))$ vanishes for all curves of genus 2 or less. □

Remark 3.1.6. Although we will be looking at some cases with negative genuses, the only curves of non-negative genuses we will be studying are lines, plane conics, plane cubics, and twisted cubics. These all have genus less than 2, so we know the Hilbert schemes corresponding these curves have to be smooth *at a point parametrizing one of these curves*. In short, we have an accurate measure of the dimension of the Hilbert scheme at $[C]$.

3.2 $\mathcal{H}_{1,0,3}$

Let us first look at the simplest 1-dimensional subschemes: the lines in $\mathbf{P}^3$. The Hilbert polynomial of a line in $\mathbf{P}^3$ is $P_L(m) = m + 1$. This can be seen by looking at the Hilbert function for a very large m, as in Section 2.3. So we are interested in $\mathcal{H}_{1,0,3}$. Originally we had mentioned that the Hilbert schemes could be viewed as a generalization of the Grassmannian varieties. Thus our instinct tells us that the Hilbert scheme corresponding to the lines in $\mathbf{P}^3$ should agree with $G(1,3)$. This turns out to be true, and we prove the following theorem.

Theorem 3.2.1. *The Hilbert scheme $\mathcal{H}_{1,0,3}$ parametrizing lines in $\mathbf{P}^3$ is isomorphic to $G(1,3)$. No other subschemes are parametrized by $\mathcal{H}_{1,0,3}$.*

The first thing we are going to do is to convince ourselves that the closed subschemes of $\mathbf{P}^3$ having Hilbert polynomial $t + 1$ are indeed lines and only lines.

Lemma 3.2.2. *Let $C \in \mathbf{P}^3$ be a subscheme having Hilbert polynomial $m + 1$. Then C is a line.*

Proof. Let $P_C(m) = m + 1$ for $C \subset \mathbf{P}^3$. A priori we know that the degree of C is 1 since the coefficient of the leading term is 1. Given this, the question is, are all degree 1 curves in $\mathbf{P}^3$ (or even in $\mathbf{P}^r$), indeed, lines? The answer is yes. Take any two distinct points of the reduced C_{red}, and consider a hyperplane that passes through both of them. By Bezout's theorem, this hyperplane must contain a component of C_{red} since it cannot intersect transversely. Now consider the intersection of all such hyperplanes. Since the intersection of hyperplanes is a line, we now have a line that contains a component of C_{red}. And since this line is irreducible it coincides with C_{red}. But we know then that the Hilbert polynomial of that line ($\cong \mathbf{P}^1$) would have to be $m + 1$ already, and C must therefore coincide. □

Now we are ready to show that $\mathcal{H}_{1,0,3} \cong G(1,3)$. The first proof we give is algebraic and is necessarily thorough. After this example, we will be involving more geometry. Recall that Grassmannians are universal schemes (for k-dimensional subspaces of $\mathbf{P}^r$). So $\mathcal{G} := G(1,3)$ comes equipped with a universal rank 2 subbundle $\mathcal{S}_{\mathcal{G}} \subset \mathcal{O}_{\mathcal{G}}^4$ whose projectivization is the universal line.

On the other hand, suppose we are given any family of lines $\phi : \mathcal{X} \to B$ in $\mathbf{P}^3$. Of course, by this we really mean the following diagram,

$$\begin{array}{ccccc} \mathcal{X} & \overset{i}{\hookrightarrow} & \mathbf{P}^3 \times_S B & \xrightarrow{\pi_P} & \mathbf{P}^3 \\ & \searrow^{\phi} & \downarrow{\scriptstyle \pi_B} & & \downarrow{\scriptstyle p} \\ & & B & \xrightarrow{g} & S \end{array} \tag{3.1}$$

where $\pi_B = g^*p$. We have an exact sequence:

$$0 \to \mathcal{I}_{\mathcal{X}} \to \mathcal{O}_{\mathbf{P}^3_B} \to \mathcal{O}_{\mathcal{X}} \to 0.$$

Since $\mathcal{O}_{\mathbf{P}^3_B}$ and $\mathcal{O}_{\mathcal{X}}$ are flat over B, $\mathcal{I}_{\mathcal{X}}$ must also be B-flat. We twist it once by the pullback of $\mathcal{O}_{\mathbf{P}^3}(1)$ and the exactness is preserved:

$$0 \to \mathcal{I}_{\mathcal{X}} \otimes \pi_P^*\mathcal{O}_{\mathbf{P}^3}(1) \to \pi_P^*\mathcal{O}_{\mathbf{P}^3}(1) \to \mathcal{O}_{\mathcal{X}} \otimes \pi_P^*\mathcal{O}_{\mathbf{P}^3}(1) \to 0.$$

We write this as

$$0 \to \mathcal{I}_{\mathcal{X}}(1) \to \mathcal{O}_{\mathbf{P}^3_B}(1) \to \mathcal{O}_{\mathcal{X}}(1) \to 0,$$

as $\mathcal{O}_{\mathbf{P}^3_B}$-modules. Since $\mathcal{O}_{\mathcal{X}}(1)$ is flat over B, we can consider this short exact sequence over the fibers. Namely, choose $b_C \in B$ corresponding to the line C, and we have

$$0 \to \mathcal{I}_C(1) \to \mathcal{O}_{\mathbf{P}^3}(1) \to \mathcal{O}_C(1) \to 0.$$

In effect, what we are doing here is to tensor the exact sequence with $\mathcal{O}_C$ over $\mathcal{O}_B$; and the exactness is preserved because $\mathcal{O}_{\mathcal{X}}(1)$ is flat over $\mathcal{O}_B$.

Lemma 3.2.3. $h^1(C, \mathcal{I}_C(1)) = 0$.

Proof. Taking the cohomology of this sequence, we get

$$0 \to H^0(C, \mathcal{I}_C(1)) \to H^0(C, \mathcal{O}_{\mathbf{P}^3}(1)) \to H^0(C, \mathcal{O}_C(1))$$
$$\to H^1(C, \mathcal{I}_C(1)) \to H^1(C, \mathcal{O}_{\mathbf{P}^3}(1)) \to H^1(C, \mathcal{O}_C(1)) \to \cdots$$

The idea here is that over a geometric point the fiber is $\mathbf{P}^1 \hookrightarrow \mathbf{P}^3$, and so we know how to calculate some of these cohomology groups already. The higher cohomology groups of $\mathcal{O}_{\mathbf{P}^3}(1)$ and of $\mathcal{O}_C(1)$ are 0. The spaces of global sections of $\mathcal{O}_{\mathbf{P}^3}(1)$ and $\mathcal{O}_C(1)$ have dimensions 4 and 2, respectively. In addition, $h^0(C, \mathcal{I}_C(1)) = 2$ since it corresponds to the linear forms in $\mathbf{P}^3$ which vanish on $\mathbf{P}^1$. Since 4=2+2, we conclude that $H^1(C, \mathcal{I}_C(1)) = 0$ for each C that is a fiber. □

In addition, pushing the original short exact sequence via π_B we get the following long exact sequence of higher direct image sheaves:

$$0 \to (\pi_B)_*\mathcal{I}_{\mathcal{X}}(1) \to (\pi_B)_*\mathcal{O}_{\mathbf{P}^3_B}(1) \to (\pi_B)_*\mathcal{O}_{\mathcal{X}}(1) \to R^1(\pi_B)_*\mathcal{I}_{\mathcal{X}}(1) \cdots$$

We can use Theorem 2.4.7.a) and Lemma 3.2.3 to conclude that all fibers of $R^1(\pi_B)_*\mathcal{I}_{\mathcal{X}}(1)$ vanish; hence $R^1(\pi_B)_*\mathcal{I}_{\mathcal{X}}(1) = 0$. We now have the exactness of the push-forward sequence:

$$0 \to (\pi_B)_*\mathcal{I}_{\mathcal{X}}(1) \to (\pi_B)_*\mathcal{O}_{\mathbf{P}^3_B}(1) \to (\pi_B)_*\mathcal{O}_{\mathcal{X}}(1) \to 0.$$

Again by Theorem 2.4.7.c), the middle term $(\pi_B)_*\mathcal{O}_{\mathbf{P}^3_B}(1) \cong g^*p_*\mathcal{O}_{\mathbf{P}^3}(1)$, which is locally free. We have the following sequences:

$$\begin{array}{ccccccc} \cdots \longrightarrow & (\pi_B)_*\mathcal{O}_{\mathbf{P}^3_B}(1) \otimes \kappa(b_C) & \xrightarrow{h_1} & (\pi_B)_*\mathcal{O}_{\mathcal{X}}(1) \otimes \kappa(b_C) & \longrightarrow 0 \\ & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & \\ \cdots \longrightarrow & H^0(C, \mathcal{O}_{\mathbf{P}^3}(1)) & \xrightarrow{h_2} & H^0(C, \mathcal{O}_C(1)) & \longrightarrow 0 \end{array} \tag{3.2}$$

The first map p_1 is surjective because $(\pi_B)_*\mathcal{O}_{\mathbf{P}^3_B}(1) \otimes \kappa(b_C)$ is a rank 4 module, and $h^0(C, \mathcal{O}_{\mathbf{P}^3}(1)) = 4$. h_2 is surjective because C is $\mathbf{P}^1 \hookrightarrow \mathbf{P}^3$. By the commutative diagram, p_2 is a surjection. But then by Theorem 2.4.7.b), $(\pi_B)_*\mathcal{O}_{\mathcal{X}}(1)$ is locally free. Since $(\pi_B)_*\mathcal{O}_{\mathbf{P}^3_B}(1)$ is locally a sum of $(\pi_B)_*\mathcal{I}_{\mathcal{X}}(1)$ and $(\pi_B)_*\mathcal{O}_{\mathcal{X}}(1)$, we conclude $(\pi_B)_*\mathcal{I}_{\mathcal{X}}(1)$ is locally free. In addition, we have already seen that all the fibers of $(\pi_B)_*\mathcal{I}_{\mathcal{X}}(1)$ are of rank 2. In other words, $(\pi_B)_*\mathcal{I}_{\mathcal{X}}(1)$ is a locally free sheaf of rank 2. Likewise the cokernel of the sequence is also locally free and of rank 2.

Taking the dual of this sequence and letting $\mathcal{S}_B := ((\pi_B)_*\mathcal{O}_{\mathcal{X}}(1))^\vee$, we have

$$0 \to \mathcal{S}_B \to ((\pi_B)_*\mathcal{O}_{\mathbf{P}^3_B}(1))^\vee \to ((\pi_B)_*\mathcal{I}_{\mathcal{X}}(1))^\vee \to 0,$$

which shows that we can actually think of our object as being cut by a rank 2 subbundle, whose cokernel is also locally free of rank 2. That is, given any family $\phi : \mathcal{X} \to B$ of lines in $\mathbf{P}^3$, we can realize it by a rank 2 subbundle $\mathcal{S}_B \subset ((\pi_B)_*\mathcal{O}_{\mathbf{P}^3_B}(1))^\vee (\cong \mathcal{O}_B^4)$. In sum, the functor Hilb^{t+1} is the same thing as the specification of rank 2 subbundles of $\mathcal{O}_B^4$. Since $\mathcal{G}$ is universal, we can realize this subbundle as the pullback of the universal subbundle by a unique morphism $\psi : B \to \mathcal{G}$. Now does the projectivization of this subbundle cut out the original family? To answer this question we ask whether $((\pi_B)^*(\pi_B)_*\mathcal{I}_{\mathcal{X}}(1))(-1) \to \mathcal{I}_{\mathcal{X}}$ is surjective. To check that a map between two coherent sheaves is surjective, it is sufficient to check the surjectivity on fibers. Again the idea is that these fibers are all lines. So is the induced map

$$((\pi_B)^*(\pi_B)_*\mathcal{I}_C(1))(-1) \to \mathcal{I}_C$$

surjective? On the left hand side we are taking all the linear polynomials from the ideal sheaf of C, and we are asking if they map onto via pullback to the ideal sheaf of C. But since C is a line, the linear polynomials in the ideal sheaf of C are the only things that cut out the line. To rephrase, what we are saying is that if we take the linear relations that cut out a line in $\mathbf{P}^3$ (or more formally, given the ideal sheaf we look at the subsheaf generated by the linear polynomials of the original ideal sheaf), then those linear relations are indeed the only relations that cut out the line! Therefore the induced map is surjective, and likewise

$$((\pi_B)^*(\pi_B)_*\mathcal{I}_{\mathcal{X}}(1))(-1) \to \mathcal{I}_{\mathcal{X}}$$

is surjective. This shows that given a family of lines, we only need to restrict our attention to the linear relations on that family of lines since their projectivization would give the original family. Thus $\pi_{B*}\mathcal{I}_X(1)$ does cut out $\mathcal{X}$.

We also need to check that the map going from $\mathcal{G}$ to B then back to $\mathcal{G}$ is an isomorphism. But this case is a bit simpler. Given a rank 2 submodules of a rank 4 modules, we are looking at all of the linear polynomials that vanish on the rank 2 submodule. These necessarily cut out a family of lines in $\mathbf{P}^3$, so we get a map from $\mathcal{G} \longrightarrow B$. We also check that going through the same line of reasoning and by way of our construction, we recover the same submodule. (Going back, we have subschemes of $\mathbf{P}^3$, and ask which polynomials vanish on the subschemes. The annihilator of these polynomials inside the rank 4 submodule is a rank 2 submodule.) This shows an explicit isomorphism between B and $\mathcal{G}$, and Theorem 3.2.1 is proved.

Having gone through all the cohomology once, we will not be dealing with similar details for the rest of the paper. There is, however, a more geometrically intuitive way of seeing why $\mathcal{H}_{1,0,3} \cong \mathcal{G}$. Once we convince ourselves that the only schemes having $t+1$ as the Hilbert polynomial are lines, we have a set-theoretic map between these two spaces. Namely, given a point in $\mathcal{G}$ corresponding to a line in $\mathbf{P}^3$, we can map it to the point in $\mathcal{H}_{1,0,3}$ corresponding to that same line. Furthermore since there exists a universal line $\mathcal{X} \hookrightarrow \mathbf{P}^3 \times \mathcal{G}$ which is flat over $\mathcal{G}$, by the universal property of the Hilbert scheme we have a morphism $f : \mathcal{G} \longrightarrow \mathcal{H}_{1,0,3}$, which is a bijection. By Remark 3.1.6, $\mathcal{H}_{1,0,3}$ is reduced and smooth. Since $G(1,3)$ is also smooth, Zariski's Main Theorem applies, and f is an isomorphism.

One thing worth noting is that even without the machinery of Theorem 3.1.4, there is an alternative way of seeing why $\mathcal{H}_{1,0,3}$ is smooth. We know by the bijective morphism that $\dim_{[C]} \mathcal{H}_{1,0,3} = 4$. In addition, using Example 2.1.5, we can calculate the dimension of the tangent space of $\mathcal{H}_{1,0,3}$. We check that for a line $L \cong \mathbf{P}^1 \subset \mathbf{P}^3$,

$$\dim T_L\mathcal{H}_{1,0,3} = h^0(L, \mathcal{N}_{L/\mathbf{P}^3}) = h^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)) = 2 + 2 = 4 = \dim_{[C]} \mathcal{H}_{1,0,3}.$$

3.3 $\mathcal{H}_{1,-1,3}$

We could ask what the parameter space for the space of a line and a point would be. We first need to parametrize all the lines and this job is taken care of by $\mathcal{G}$. We also need to worry about the points of $\mathbf{P}^3$. We prove the following theorem.

Theorem 3.3.1. *A general point of $\mathcal{H}_{1,-1,3}$ parametrizes a line and a point in $\mathbf{P}^3$ and its specialization. It has dimension 7 and is isomorphic to*

$$\mathrm{Bl}_\Sigma(\mathcal{G} \times \mathbf{P}^3),$$

where $\Sigma = \{([L], p) | p \in L\}$ is the incidence correspondence, also known as the universal line.

Lemma 3.3.2. *If $C \in \mathbf{P}^3$ is a subscheme such that $P_C(m) = m + 2$, then C is a line and a point, or their specialization.*

Proof. Again, as in Lemma 3.2.2, we could look at C_{red} which is the reduced scheme. By the same reasoning, C_{red} is a line, which has the Hilbert polynomial $m + 1$. So C must contain a point, either embedded or separate. □

We prove the isomorphism via following four steps:

1. We exhibit a bijective set-theoretic map ρ from $\mathcal{A} := \mathrm{Bl}_\Sigma(\mathcal{G} \times \mathbf{P}^3)$ to $\mathcal{H}_{1,-1,3}$.

2. We show that $\mathcal{H}_{1,-1,3}$ is smooth. Notice that Theorem 3.1.4 does *not* apply here because we have not shown that $\mathcal{O}_C(1)$ is nonspecial for the case with genus -1.

3. We show that ρ is continuous and that it extends to a morphism.

4. We show that ρ is an isomorphism.

Step 1. The set-theoretic map is clear away from the exceptional divisor. We map a point corresponding to $([L], p)$ to the point of $\mathcal{H}_{1,-1,3}$ which parametrizes this pair. It is bijective here. A point on the exceptional divisor corresponds to a nonzero section of the normal bundle, and so we map it to the point on $\mathcal{H}_{1,-1,3}$ parametrizing $([L'], p')$ such that p' is embedded in L' with the specified normal direction. Since the subschemes parametrized by $\mathcal{H}_{1,-1,3}$ are precisely these, and by the argument included in Example 2.2.3 the map is a bijection.

Step 2. The family is flat away from the incidence correspondence, and so it is a rational map (actually a morphism) away from the incidence correspondence. Since the closure of $\mathcal{G} \times \mathbf{P}^3 - \Sigma$ is the whole space, we conclude that the image of $\mathcal{G} \times \mathbf{P}^3$ is an open dense subset in $\mathcal{A}$. Since the dimension of an open sense subset is equal to the dimension of the whole space [16, Ex.3.20], the dimension of $\mathcal{A}$ is the dimension of $\mathcal{G} \times \mathbf{P}^3$ which is 7.

In addition, notice that the normal bundle calculation from Example 2.1.12 tells us that $\dim \mathcal{T}_{[C]}\mathcal{H}_{1,-1,3} = 7$, which shows that $\mathcal{H}_{1,-1,3}$ is smooth at all points.

Step 3. To see that the map is continuous, we observe that all the curves of degree 1 and genus -1 fall into two different orbits under the general linear group action (by PGL_4). We will explain this in much more detail in Chapter 4. For now we note that if Orbit A contains all the lines with separate points and Orbit B contains all the lines with embedded points, since the map is continuous at all points corresponding to Orbit A, and Orbit B is of codimension 1, ρ is continuous by the following lemma.

Lemma 3.3.3. *If $T : X \longrightarrow Y$ is a birational transformation of projective varieties, and if X is normal, then the fundamental points of T form a closed subset of codimension greater than 2.*

Proof. The following proof is from [16, V.5.1]. If $P \in X$ is a point of codimension 1, then $\mathcal{O}_{P,X}$ is a discrete valuation ring. Since T is defined at the generic point of X, and Y is projective, hence proper, it follows from the valuative criterion of properness that T is also defined at P. □

Here T is the inverse map of ρ. We have already seen that $\mathcal{H}_{1,-1,3}$ is smooth, hence normal. Finally we conclude by Corollary 2.4.9 that ρ is a morphism.

Step 4. Since $\mathcal{A}$ is a blow up of a smooth variety along a smooth subvariety, it is smooth. Using Zariski's Main Theorem, we conclude that ρ is an isomorphism. This completes the proof of Theorem 3.3.1.

Without discussing further cases of $\mathcal{H}_{1,-n,3}$ we proceed on to curves of degree 2.

3.4 $\mathcal{H}_{2,0,3}$

The next simplest curves are plane conics (in $\mathbf{P}^3$, of course). The Hilbert polynomial of a plane conic is $P_C(m) = 2m + 1$.

Theorem 3.4.1. *$\mathcal{H}_{2,0,3}$ parametrizes plane conics (smooth and singular, including planar double lines). No other subschemes are parametrized by $\mathcal{H}_{2,0,3}$. It is of dimension 8 and is isomorphic to the $\mathbf{P}^5$-bundle $\mathbf{P}\mathrm{Sym}^2\mathcal{S}^*$ where $\mathcal{S}$ is the universal subbundle of $\mathbf{P}^{3*}$.*

Here we are lucky because 0 is the maximum genus for any degree 2 curves in $\mathbf{P}^3$. After introducing a new terminology, we proceed as before.

Definition 3.4.2. Given a curve $C \in \mathbf{P}^3$, we define the *reduced curve* C_{red} to be the union of all 1-dimensional irreducible components of C. We note that $P_{C_{red}}(m) \leq P_C(m)$ for all positive m.

Lemma 3.4.3. *If $C \in \mathbf{P}^3$ has Hilbert polynomial $2m+1$ then C is a plane conic or its specialization.*

Proof. We know the degree of C is 2. Consider C_{red} whose degree can be either 1 or 2.

i) If it is 2, and irreducible, then by Bezout's theorem any hyperplane containing any three points of it must contain the entire C_{red}. This case C_{red} is a degree 2 curve contained in a plane – it is a plane conic. But since the Hilbert polynomial of a plane conic is already $2m+1$, $C_{red} = C$.

ii) If the degree is 2, but C_{red} is reducible, then it must consist of two components each having degree 1. By a similar argument as in Lemma 3.2.2, each of them must be a line. In this case, they must be incident lines since the Hilbert polynomial of two skew lines is $2m+2$, which is already too big; and this cannot be a double line otherwise the degree of C_{red} would have been 1. This is a singular plane conic with $2m+1$ already, so $C = C_{red}$.

iii) If the degree of C_{red} is 1, then it is a line. Since C has degree 2, we conclude this must be a scheme with multiplicity 2 two structure whose support is a line. This case was discussed thoroughly in Section 2.3, and we see that the only scheme with Hilbert polynomial $2m+1$ is a planar double line, which is a specialization of a plane conic. □

This indicates that the Hilbert scheme does not consist of any ghost components.[1] More specifically, we only need to consider parametrizing conics in $\mathbf{P}^3$ Our proof involves constructing a map from the trivial bundle, which cuts out plane conics in $\mathbf{P}^3$. By showing that this exhibits a flat family, we shall use the universal property to construct a morphism to the Hilbert scheme. Although it is standard to look at the ambient space of the universal family as $\mathcal{H}_{2,0,3} \times \mathbf{P}^3$, since plane conics are degenerate (i.e. they are contained in planes), in order to package this information, we look at the ambient space in terms of the universal planes of $\mathbf{P}^3$. The ambient space we shall consider is $\mathbf{P}(\mathrm{Sym}^2(\mathcal{S}^*)) \times_{\mathbf{P}^{3*}} \mathbf{P}(\mathcal{S})$, since it is more convenient to see the flatness of the family this way.

Before we begin the proof, we include a brief overview of projective cones and bundles. This material comes partly from [7, B.5]. Given a vector bundle $p : E \longrightarrow X$, there is a canonical surjection $p^*E^\vee \longrightarrow \mathcal{O}_E(1)$ on $\mathbf{P}(E)$. Here by $\mathbf{P}(E)$ we mean $\mathrm{Proj}(\mathrm{Sym}E^\vee)$. This gives an embedding

$$\mathcal{O}_E(-1) \longrightarrow p^*E.$$

We call $\mathcal{O}_E(-1)$ the *universal* or tautological subbundle.

In addition, given a variety X and a vector bundle E on X (of rank n), and given k, the Grassmannian bundle

$$p : G(k, E) \longrightarrow X,$$

which parametrizes k-dimensional subspaces of the fibers of E, carries a universal rank k subbundle S of p^*E and a universal rank n-k quotient Q of p^*E:

$$0 \longrightarrow S \longrightarrow p^*E \longrightarrow Q \longrightarrow 0.$$

Again, S is just the tautological bundle (i.e., if L is a k-dimensional subspace of the fiber of E over a point x, then the fiber of S at the point of L is simply L). But if we take the dual of the sequence above, we may identify $Q^\vee$ as a rank n-k subbundle of $p^*(E^\vee)$. This induces a map $G(k, E) \longrightarrow G(n-k, E^\vee)$. This map is naturally an isomorphism. So we treat them as the same space.

We utilize this concept to our problem. A conic is an intersection of a plane and a quadric surface. In particular, given a conic, the plane that contains the conic is specified. We therefore have the following set-theoretic map:

$$\rho : \mathcal{H}_{2,0,3} \longrightarrow \mathbf{P}^{3*},$$

[1] Recall that the *ghost component* of a Hilbert scheme, if exists, is the component whose general point parametrizes a curve of higher genus and extra zero-dimensional subschemes.

where ρ maps the point $[C]$ on $\mathcal{H}_{2,0,3}$ corresponding to the conic C to the point $[\lambda_C] \in \mathbf{P}^{3*}$ corresponding to the plane $\lambda_C \subset \mathbf{P}^3$, on which lies C. What do we know about this map? Suppose we take $[\lambda] \in \mathbf{P}^{3*}$. The fiber is the projective space corresponding to the quotient of the space of quadratic polynomials by the subspace of quadratic polynomials which vanish on the plane $\lambda \subset \mathbf{P}^3$ (in other words, the space of quadratic polynomials over the plane λ). This is $\mathbf{P}(\mathrm{Sym}^2\mathbf{P}([\lambda])^*) \cong \mathbf{P}^5$.

Now we consider the rank 3 universal subbundle $\mathcal{S} \longrightarrow \mathbf{P}^{3*}$. By this we mean the subbundle that satisfies the following exact sequence

$$0 \to \mathcal{S} \to \mathcal{O}_{\mathbf{P}^{3*}}^{\oplus 4} \to \mathcal{O}_{\mathbf{P}^{3*}}(1) \to 0,$$

whose projectivization $\mathbf{P}(\mathcal{S})$ is the universal hyperplane sitting inside $\mathbf{P}^3 \times \mathbf{P}^{3*}$. After all, $\mathbf{P}^{3*}$ can be viewed as a Grassmannian variety (of 2-planes in $\mathbf{P}^3$.) The above sequence is the dual of the following usual sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{3*}}(-1) \longrightarrow \mathcal{O}_{\mathbf{P}^{3*}}^{\oplus 4} \longrightarrow Q \longrightarrow 0,$$

with $Q^\vee = S$. Then $\mathbf{P}\mathrm{Sym}^2\mathcal{S}^* \to \mathbf{P}^{3*}$ is the space parametrizing pairs (H, Q) where $H \subset \mathbf{P}^3$ is a hyperplane and $Q \subset \mathbf{P}H^0(H, \mathcal{O}_H(2))$ is a quadratic polynomial on H up to scaling. Now consider the commuta-

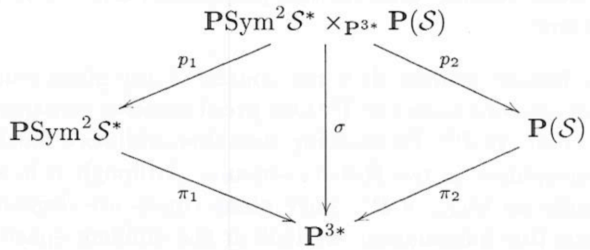

Figure 3.1: $\sigma = \pi_1 \circ p_1 = \pi_2 \circ p_2$

tive diagram shown in Figure 3.1. Notice that, since $\mathbf{P}(\mathcal{S})$ is a universal plane sitting inside $\mathbf{P}^3 \times \mathbf{P}^{3*}$, $\mathbf{P}(\mathrm{Sym}^2(\mathcal{S}^*)) \times_{\mathbf{P}^{3*}} \mathbf{P}(\mathcal{S})$ sits inside

$$\mathbf{P}(\mathrm{Sym}^2(\mathcal{S}^*)) \times_{\mathbf{P}^{3*}} (\mathbf{P}^3 \times \mathbf{P}^{3*}),$$

which is $\mathbf{P}(\mathrm{Sym}^2(\mathcal{S}^*) \times \mathbf{P}^3$ by "cancellation of fiber product." Pulling back from $\mathbf{P}^{3*}$ we have the following two maps,

$$\begin{array}{rcl} \mathcal{O}_{\mathbf{P}_{\mathrm{Sym}^2\mathcal{S}^*}}(-1) & \to & \pi_1^*\mathrm{Sym}^2\mathcal{S}^* \\ \mathcal{O}_{\mathbf{P}(\mathcal{S})}(-1) & \to & \pi_2^*\mathcal{S} \end{array}$$

which can, then, be pulled back once more to yield

$$\begin{array}{rcl} p_1^*\mathcal{O}_{\mathbf{P}_{\mathrm{Sym}^2\mathcal{S}^*}}(-1) & \to & p_1^*\pi_1^*\mathrm{Sym}^2\mathcal{S}^* = (\pi_1 \circ p_1)^*\mathrm{Sym}^2\mathcal{S}^* = \sigma^*\mathrm{Sym}^2\mathcal{S}^* \\ p_2^*\mathcal{O}_{\mathbf{P}(\mathcal{S})}(-1) & \to & p_2^*\pi_2^*\mathcal{S} = (\pi_2 \circ p_2)^*\mathcal{S} = \sigma^*\mathcal{S}. \end{array}$$

In general, given a vector bundle $E \longrightarrow S$, we have the induced vector bundle $\mathrm{Sym}^2 E \longrightarrow \mathrm{Sym}^2 S$. And since σ^* commutes with tensor products, we now have the map to the trivial bundle of $\mathbf{P}\mathrm{Sym}^2\mathcal{S}^* \times_{\mathbf{P}^{3*}} \mathbf{P}(\mathcal{S})$ as in Figure 3.2. We are interested in taking a point in $\mathbf{P}(\mathcal{S})$, which is a vector in a two plane, and then evaluating the universal quadratic equation. So $\mathbf{P}\mathrm{Sym}^2\mathcal{S}^*$, which is the space parametrizing quadratic polynomials on $\mathbf{P}^2$'s up to scaling, takes the universal quadratic polynomial on that $\mathbf{P}^2$, and take the vector on $\mathbf{P}^2$ (on which we want to evaluate). But to do that we have to square the vector – if we identify a polynomial as an element of $\mathrm{Sym}^d\mathcal{S}^*$, then in order to evaluate it on a vector, we need to take the d-th power of the vector. This is precisely why we look at

$$p_1^*\mathcal{O}_{\mathbf{P}_{\mathrm{Sym}^2\mathcal{S}^*}}(-1) \otimes p_2^*\mathcal{O}_{\mathbf{P}(\mathcal{S})}(-2).$$

$$\begin{array}{ccc} p_1^*\mathcal{O}_{\mathbf{P}_{\mathrm{Sym}^2\mathcal{S}^*}}(-1)\otimes p_2^*\mathcal{O}_{\mathbf{P}(\mathcal{S})}(-2) & \longrightarrow & \sigma^*(\mathrm{Sym}^2\mathcal{S}^*\otimes \mathrm{Sym}^2(\mathcal{S})) \\ & \searrow^{\psi} & \downarrow \\ & & \mathcal{O} \end{array}$$

Figure 3.2: Map to the Trivial Bundle

The vertical map is a trace map, since we can rewrite $\mathrm{Sym}^2\mathcal{S}^*\otimes\mathrm{Sym}^2(\mathcal{S})$ as $(\mathrm{Sym}^2\mathcal{S})^*\otimes\mathrm{Sym}^2\mathcal{S}$. Thus the family of conics is the zero locus of the global sections of the dual of ψ.

$$\psi^\vee : \mathcal{O}\longrightarrow p_1^*\mathcal{O}_{\mathbf{P}_{\mathrm{Sym}^2\mathcal{S}^*}}(1)\otimes p_2^*\mathcal{O}_{\mathbf{P}(\mathcal{S})}(2) \qquad (3.3)$$

A more intuitive way to look at this map is to interpret it as ψ twisted up by $p_1^*\mathcal{O}_{\mathbf{P}_{\mathrm{Sym}^2\mathcal{S}^*}}(1)\otimes p_2^*\mathcal{O}_{\mathbf{P}(\mathcal{S})}(2)$ – this way we see that the zero locus of ψ remains the same.

Locally speaking, in a small neighborhood of $p_1^*\mathcal{O}_{\mathbf{P}_{\mathrm{Sym}^2\mathcal{S}^*}}(1)\otimes p_2^*\mathcal{O}_{\mathbf{P}(\mathcal{S})}(2)$ we can choose a basis $e_1,...,e_m$ and take this map as a set of functions $f_1,...,f_n$. We want to prove that this exhibits a flat family of plane conics. We recall the following two propositions:

Proposition 3.4.4. *Let R be a locally Cohen-Macaulay ring. If $I=(x_1,...,x_n)$ is an ideal generated by n elements in a locally Cohen-Macaulay ring R such that* codim $I=n$, *the largest possible value, then R/I is a Cohen-Macaulay ring.*

Reference. [4, Theorem 18.13]. □

Proposition 3.4.5. *Let $f:Y\longrightarrow X$ be a morphism of k-schemes. Set $\dim X=n$, $\dim Y=m$, and suppose that the following conditions hold: (1) X is regular; (2) Y is Cohen-Macaulay; (3) f takes closed points of Y into closed points of X (this holds for example if f is proper); (4) for every closed point $x\in X$ the fibre $f^{-1}(X)$ is $(m-n)$-dimensional (or empty). Then f is flat.*

Reference. [21, Corollary to Theorem 23.1.] □

Cohen-Macaulay is a generalized notion of local complete intersections. What is going on? In general, if we have a smooth variety X (or even just locally Cohen-Macaulay), a rank n vector bundle E on X, and a global section s of E, the zero locus Y of s has codimension at most n. And if Y has codimension n, then it is also locally Cohen-Macaulay. So in our case, since C is the zero locus of a section of a line bundle, it is locally Cohem-Macaulay if and only if it has codimension 1 (which, of course, it does). So we know C is locally Cohen-Macaulay. Since Y is smooth, to check that $C\longrightarrow Y$ is flat, it suffices to check that the fiber dimension of $C\longrightarrow Y$ is always 1. Since the projection map $Y\times_{\mathbf{P}^{3*}}\mathbf{P}(\mathcal{S})\longrightarrow Y$ has fiber dimension 2, either the dimension of a fiber of $C\longrightarrow Y$ is 1 or else the entire fiber of $Y\times_{\mathbf{P}^{3*}}\mathbf{P}(\mathcal{S})\longrightarrow Y$ over that point is in C. But this implies the quadratic polynomial defining C on this fiber vanishes identically on $\mathbf{P}(\mathcal{S})$. This contradicts the definition of Y as the projective bundle of nonzero quadratic polynomials on $\mathbf{P}(\mathcal{S})$.

There is a less formal way to see this also. A weaker version of the combination of the above two propositions basically says that if the curve is locally a complete intersection, in order to check the flatness of the family we just need to check that the dimension does not change. But since all conics are clearly 1-dimensional subschemes, what we have is indeed a flat family of conics $\mathcal{C}\to\mathbf{P}\mathrm{Sym}^2\mathcal{S}^*$. Furthermore, the universal property of the Hilbert scheme guarantees a morphism $\psi:\mathbf{P}\mathrm{Sym}^2\mathcal{S}^*\to\mathcal{H}_{2,0,3}$. Once again, we have a morphism which is set-theoretically a bijection – it is a bijection because no two distinct planes contain a conic, and given the plane and the coordinate system there is a unique degree 2 polynomial up to scaling that cuts out the conic. Again Remark 3.1.6 tells us that $\mathcal{H}_{2,0,3}$ is smooth. This can also be verified by the normal bundle calculation from Example 2.1.6 since $\dim T_C\mathcal{H}_{2,0,3}=h^0(C,\mathcal{N}_{C/\mathbf{P}^3})=h^0(C,\mathcal{O}_{\mathbf{P}^1}(4)\oplus\mathcal{O}_{\mathbf{P}^1}(2))=5+3=8$. By Zariski's Main Theorem, we have an isomorphism between $\mathcal{H}_{2,0,3}$ and the $\mathbf{P}^5$-bundle $\mathbf{P}\mathrm{Sym}^2\mathcal{S}^*$.

Remark 3.4.6. That $\mathcal{H}_{2,0,3}$ is of dimension 8 makes sense intuitively. If we are talking about a family of plane conics, we would have degree 3 freedom of choosing the plane and degree 5 freedom of choosing the conic on the specified plane. But then this scheme also contains any sort of specializations of plane conics. For instance, two incident lines or double lines arise this way. The space of two incident lines here forms a 7-dimensional subscheme. Meanwhile the space of double line in this case has dimension 5 because we are only concerned with the double lines which lie on a plane. We will come back to this idea in Chapter 4.

3.5 $\mathcal{H}_{2,-1,3}$

Let us now attack $\mathcal{H}_{2,-1,3}$. This is where things get more interesting since -1 is *not* the maximum genus of the degree 2 curves in $\mathbf{P}^3$; instead 0 is the maximum genus.[2] So then what types of curves do get parametrized by $\mathcal{H}_{2,-1,3}$? These should have $P_C(m) = 2m + 2$. Two types of curves readily come to our minds. One is a pair of skew lines having the Hilbert polynomial $(m + 1) + (m + 1) = 2m + 2$. The other is the union of a plane conic and a point, which would have $(2m+1)+1 = 2m+2$. Already we are stepping into an unfamiliar territory of the Hilbert schemes. For instance, it is not known exactly what each component looks like. We have some partial but fairly complete descriptions of $\mathcal{H}_{2,-1,3}$ which we exhibit in this section. We prove the following theorem.

Theorem 3.5.1. *Let $\mathcal{H} := \mathcal{H}_{2,-1,3}$ be the Hilbert scheme parametrizing two skew lines in $\mathbf{P}^3$.*

a) $\mathcal{H} = \mathcal{H}' \cup \mathcal{H}''$, where a general point of $\mathcal{H}'$ parametrizes a pair of skew lines and a general point of $\mathcal{H}''$ parametrizes a plane conic and a point.

b) Let $\overline{\mathcal{H}'} := \mathrm{Bl}_{\triangle}\mathrm{Sym}^2\mathcal{G}$ where $\triangle$ is the diagonal of $\mathcal{G} \times \mathcal{G}$. This scheme is smooth and has dimension 8, and there is a bijective morphism from $\overline{\mathcal{H}'}$ to $\mathcal{H}'$.

c) Let $\overline{\mathcal{H}''} := \mathrm{Bl}_{\Sigma}(\mathcal{H}_{2,0,3} \times \mathbf{P}^3)$ where $\Sigma = \{([C], p) | p \in C\}$ is the incidence correspondence (or the universal curve) of $\mathcal{H}_{2,0,3} \times \mathbf{P}^3$. This scheme is smooth and has dimension 11, and there is a bijective morphism from $\overline{\mathcal{H}''}$ to $\mathcal{H}''$.

d) If $\mathcal{H}'$ and $\mathcal{H}''$ are individually smooth, then we have $\mathcal{H}' \cong \overline{\mathcal{H}'}$ and $\mathcal{H}'' \cong \overline{\mathcal{H}''}$.

Remark 3.5.2. Part d) is obvious, but is mentioned for courtesy so that in the case that $\mathcal{H}'$ and $\mathcal{H}''$ are indeed smooth we have a better sense of the individual shapes.

Although $\mathrm{Sym}^2\mathcal{G}$ is not smooth, the blow up along the diagonal is smooth. Also notice that Σ has dimension 9 and a fiber of $\Sigma \to \mathbf{P}^3$ above a point p is the set of all conics through p. This is equal to a $\mathbf{P}^4$-bundle over the $\mathbf{P}^2$ of planes through p. This, in turn, is smooth. So $\mathcal{H}''$ is a blow up of a smooth variety along a smooth subvariety, and thus is also smooth. Since dimensions do not change under blow-up's, $\dim \overline{\mathcal{H}'} = 8$ and $\dim \overline{\mathcal{H}''} = 11$.

Lemma 3.5.3. *If $C \in \mathbf{P}^3$ has Hilbert polynomial $2m + 2$, then C is a specialization of either 1) two skew lines, or 2) a plane conic and a point.*

Proof. We know by the leading term of the Hilbert polynomial that this subscheme is of degree 2. Once again we consider C_{red} whose degree could be either 1 or 2.

i) If $\deg(C_{\mathrm{red}})$ is 2 and it is irreducible, by the same argument as in Lemma 3.3.3, C_{red} is a plane conic. Since the Hilbert polynomial is only $2m+1$ in this case, we must have an extra point, embedded or separate.

ii) If $\deg(C_{\mathrm{red}})$ is 2, and it is reducible, again C_{red} must be a union of two lines. In this case, two skew lines have Hilbert polynomial $2m + 2$ and $C_{\mathrm{red}} = C$.

iii) If $\deg(C_{\mathrm{red}})$ is 1, then C is a line with multiplicity two structure. By Section 2.3, the only such subschemes $C \in \mathbf{P}^3$ are double lines with genus -1 or genus 0 since any other double lines have Hilbert

[2] For a more comprehensive coverage of classification of curves in $\mathbf{P}^3$, refer to [16, IV. 6].

polynomials that are too large; the former is a specialization of two skew lines, and the latter must contain an extra point (hence a specialization of a plane conic and a point).

This proves Theorem 3.5.1.a). □

By Hartshorne's theorem [17], we know the two components are connected to each other (although each component might not necessarily be a connected one individually). The intersection of $\mathcal{H}'$ and $\mathcal{H}''$ would be those curves that arise as specialization of both types – it consists of a pair of incident lines with an embedded point. This embedded point arises as a flat limit of two skew lines approaching each other. As we saw in Section 2.2, this is a spatial embedded point.

In this section we analyze each of these components separately. Let us first take a look at $\mathcal{H}'$. Since the Hilbert scheme of lines in $\mathbf{P}^3$ is $\mathcal{G}$, we may intuitively guess $\mathcal{H}'$ may be $\mathrm{Sym}^2\mathcal{G}$. After all, switching the two lines should correspond the same point on the Hilbert scheme. But then we have a problem along the diagonal since a double line is not uniquely determined in projective spaces; and this is precisely why we want to blow up this scheme along the diagonal.

We claim that there exists a morphism to this component from the blow up of $\mathrm{Sym}^2\mathcal{G}$ along the diagonal. We prove this assertion by going through the following four steps:

1. Let $\mathcal{G}_2$ denote $\mathrm{Bl}_\triangle(\mathrm{Sym}^2\mathcal{G})$ and $\mathcal{G'}_2$ denote $\mathrm{Bl}_\triangle(\mathcal{G}\times\mathcal{G})$. We exhibit a set-theoretic map $\rho : \mathcal{G}_2' \longrightarrow \mathcal{H}_{2,-1,3}$.

2. We show that $\rho = \rho \circ \Sigma$ where $\Sigma : \mathcal{G}\times\mathcal{G} \longrightarrow \mathcal{G}\times\mathcal{G}$ is the involution (switching the two components of $\mathcal{G}\times\mathcal{G}$).

3. We then construct a morphism from $\mathcal{G}_2'$ to $\mathcal{G}_2$, and conclude that $\mathcal{G}_2'/\Sigma \cong \mathcal{G}_2$.

4. We finally show that the map constructed in Step 1 extends to a regular map.

Step 1. First of all, away from the diagonal, if we have two lines (by this, we mean the point $([L],[L']) \in \mathcal{G}\times\mathcal{G}$ where $[L]$ and $[L']$ are the points in $\mathcal{G}$ representing the lines L and L' in $\mathbf{P}^3$ respectively) which are skewed, we can take the union of those two skew lines as a subscheme of $\mathbf{P}^3$ and that would give us a point on $\mathcal{H}_{2,-1,3}$. If we have two intersecting lines, the scheme we want would have to have an embedded point as shown in Example 2.2.4. The embedded point is normal to the plane spanned by the two lines and this scheme has a 3-dimensional Zariski tangent space at the intersection. In addition, as there is no ambiguity of embedded points with regards to the two incident lines, we again have a map of points for ρ. Notice that this map, as we have it, is generically 2-to-1.

Now suppose we have $p \in \triangle(\mathcal{G}\times\mathcal{G})$, i.e. $p = ([L],[L])$. The fiber of $\triangle$ under the blow-up corresponds to an exceptional divisor, and the fiber over a point in $\triangle$ is isomorphic to $\mathbf{P}^1$. A double line L is equipped with a specification of a normal direction at every point. So we need to see how the points on this exceptional divisor essentially provide the data of the line L along with a section of normal direction at every point (that is, with a nonzero section of $\mathcal{N}_{L/\mathbf{P}^3}$ up to scalars).

More generally if $Y \subset X$ is any smooth subvariety, then we can consider the map from E, the exceptional divisor of $\mathrm{Bl}_Y(X)$, to Y.

$$\begin{array}{ccc} E & \subset & \mathrm{Bl}_Y(X) \\ \downarrow & & \downarrow \\ Y & \subset & X \end{array} \tag{3.4}$$

The vector bundle $E \to Y$ is the projective vector bundle $\mathbf{P}(\mathcal{N}_{Y/X})$. In particular, with our situation, we have the diagonal $\triangle \subset \mathcal{G}\times\mathcal{G}$. By Example 2.1.8, we conclude that $E_\triangle \to \triangle$ is $\mathbf{P}(\mathcal{T}_\mathcal{G})$. So $E \to Y$ corresponds precisely to the projectivization of the tangent space to the Grassmannian. But we know a priori that the tangent space to $G(1,3)$ is isomorphic to $\mathrm{Hom}(V, \mathbf{C}^4/V)$ where V is a two-dimensional vector subspace. This is enough to give a scheme structure to the double line L, since specifying the ambient plane at every point is equivalent to specifying a map

$$\phi : L = \mathbf{P}(V) \longrightarrow \mathbf{P}(\mathbf{C}^4/V).$$

There is a minor point worth mentioning here. Given elements of $\mathrm{Hom}(V, \mathbf{C}^4/V)$ we can consider rank 2 maps and rank 1 maps. With rank 2 maps, the ambient planes are indeed specified. When the maps are of rank 1, the tangent planes are constant except at the point corresponding to the kernel of this map (in the case of rank 2 maps, the kernel is trivial, and there is no point on the line corresponding to it). In this case we have an embedded point and a double line which is contained in a plane otherwise. (Refer to Example 2.2.7.)

Step 2. This point is quite trivial. By checking both away from the diagonal and on the diagonal, we see that switching the components does not have any effect on our constructed map ρ.

Step 3. We need to somehow link $\mathcal{G}'_2$ with the desired $\mathcal{G}_2$. In order to do this, we employ Theorem 2.4.10 to the following diagram:

$$\begin{array}{ccc} \mathcal{G}'_2 & \overset{\phi}{\dashrightarrow} & \mathcal{G}_2 \\ \downarrow & \overset{\sigma}{\searrow} & \downarrow \\ \mathcal{G} \times \mathcal{G} & \longrightarrow & \mathrm{Sym}^2\mathcal{G} \end{array} \tag{3.5}$$

The pre-image of $\triangle \subset \mathrm{Sym}^2\mathcal{G}$ under σ is a *Cartier divisor* since $\mathcal{G}'_2$ is smooth and $\triangle$ is of codimension 1. By the universal property of blow up, the map factors through $\mathcal{G}_2$, and there exists a morphism ϕ from $\mathcal{G}'_2 \to \mathcal{G}_2$. Having established a morphism ϕ, we want to see that $\mathcal{G}'_2/\Sigma \cong \mathcal{G}_2$. We will make the use of the following lemma:

Lemma 3.5.4. *Let X be any scheme and let H be some finite group acting on X. If $Y \subset X$ is any H-invariant subvariety, then*

$$(\mathrm{Bl}_Y X)/H \cong \mathrm{Bl}_{Y/H}(X/H).$$

Proof. Let $A := A(X)$, and consider $I_Y \subset A$. Then $\mathrm{Bl}_Y X = \cup_{a \in I_Y} U_a$ where U_a is a variety whose coordinate ring is $A[\frac{b}{a} : b \in I_Y]$, which in turn is contained in the ring of fractions of A, $K(A)$. Now we let A^H be the set of the invariant elements of A under the action of H, and denote similarly for the ideals of A.

Under that notation, consider I_Y^H. If $a \in I_Y^H$, then $a \in I_Y$. Therefore we have the following inclusion:

$$A^H[\frac{b}{a} : b \in I_Y^H] \subset (A[\frac{b}{a} : b \in I_Y])^H.$$

Now if $c \in A[\frac{b}{a} : b \in I_Y]$, we can write it as a linear combination of powers of b/a's. Since b is contained in an ideal, we can clear the denominator to write c as b'/a^n for some $n \geq 0$ and $b' \in I_Y$. Notice that if c is invariant under the action of H, then for each $h \in H$

$$h(c) = c \implies h(\frac{b'}{a^n}) = \frac{b'}{a^n} \implies \frac{b'}{a^n} = h(\frac{b'}{a^n}) = \frac{h(b')}{h(a^n)} = \frac{h(b')}{(h(a))^n} = \frac{h(b')}{a^n}$$

$$\implies h(b') = b' \implies b' \in I_Y^H \implies c = \frac{b'}{a^n} \in A^H[\frac{b}{a} : b \in I_Y^H].$$

We therefore conclude that

$$A^H[\frac{b}{a} : b \in I_Y^H] = (A[\frac{b}{a} : b \in I_Y^H])^H.$$

But the right quantity is precisely the form of the coordinate rings corresponding to the $\mathrm{Bl}_Y X$ modulo H, and the left corresponds to those arising from $\mathrm{Bl}_{Y/H} X/H$. Applying this argument locally for every U_a, we can glue together and finally reach the desired isomorphism. □

Corollary 3.5.5. $\mathcal{G}'_2/\Sigma \cong \mathcal{G}_2$.

Proof. Take $\mathbf{Z}_2$ as the involution group acting on $\mathcal{G} \times \mathcal{G}$, and the result is immediate. □

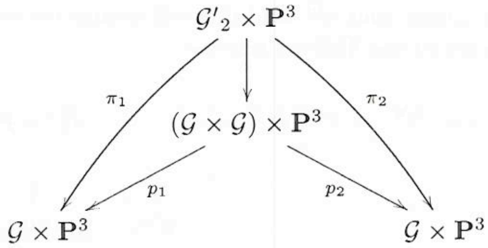

Figure 3.3: Schematic Diagram of $\mathcal{G}'_2 \times \mathbf{P}^3$

Step 4. We are now in a position to justify that the map of the points specified in Step 1 is indeed a morphism. By Corollary 2.4.9, our task of showing that ρ is a morphism now boils down to showing that ρ is continuous. In effect, what we have corresponds to the diagram in Figure 3.3, and we are considering $\pi_1^*(\mathcal{X}) \cup \pi_2^*(\mathcal{X}) \to \mathcal{G}'_2$ where $\mathcal{X} \to \mathcal{G}$ is the universal line. Away from the diagonal, this family is flat and so the map is continuous. To check that on the diagonal the map is indeed continuous, we need to look at a small neighborhood on the diagonal. This can be done by looking at a fixed line L_0 and a moving line L_t which approaches L_0. (Actually if we were to prove that the family is flat over the diagonal, then we would have to be looking at two moving lines. But here, instead, we restrict our attention to proving the continuity of the map.) We just need to check that the scheme structure agrees with the expected one (specified by our set-theoretic map ρ on the exceptional divisor). But as we saw in Example 2.2.6, the limiting scheme of $C_t = L_t \cup L_0$ preserves the information of how L_t approached L_0 to the first order. This is equivalent to reading off of the ideal of C_0 the tangent vector to the Grassmannian $\mathcal{G}$; this agrees with our set-theoretic map since we showed that $E_\triangle \longrightarrow \triangle$ is $\mathbf{P}\mathcal{T}_\mathcal{G}$. We conclude the map is continuous, which in turn is regular by 2.4.9. This shows that ρ as constructed in Step 1 is a morphism, which is generically 2-to-1 onto $\mathcal{H}'$. But since ρ is invariant under the action of Σ, we have an induced morphism

$$\mathcal{G}_2 \cong \mathcal{G}'_2/\Sigma \longrightarrow \mathcal{H}$$

by the Universal Property of Quotient. Hence we have constructed a bijective morphism between $\mathrm{Bl}_\triangle(\mathrm{Sym}^2\mathcal{G})$ and $\mathcal{H}'$. This proves Theorem 3.5.1.b).

Now we study the more complicated component $\mathcal{H}''$. Since the general member of this family is the union of a plane conic and a point, we would expect it to be some form of $\mathcal{H}_{2,0,3} \times \mathbf{P}^3$. Indeed we construct a morphism to $\mathcal{H}''$ from $\overline{\mathcal{H}''}$, the blow up of $\mathcal{H}_{2,0,3} \times \mathbf{P}^3$ along the incidence correspondence.

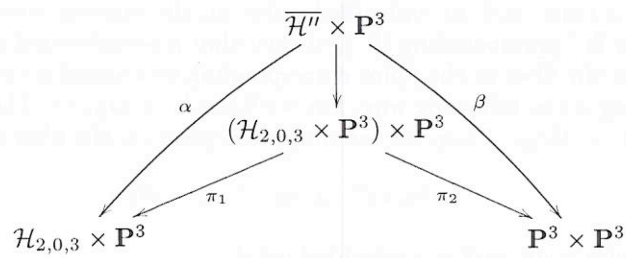

Figure 3.4: Schematic Diagram of $\mathrm{Bl}_\Sigma(\mathcal{H}_{2,0,3} \times \mathbf{P}^3)$

We begin by looking at Figure 3.4 from which we get two maps α and β from $\overline{\mathcal{H}''} \times \mathbf{P}^3$ to $\mathcal{H}_{2,0,3} \times \mathbf{P}^3$ and $\mathbf{P}^3 \times \mathbf{P}^3$, respectively. Now consider the fibers of the universal curve $\Sigma \hookrightarrow \mathcal{H}_{2,0,3} \times \mathbf{P}^3$ under α and of the universal point (otherwise known as the diagonal) $\triangle \hookrightarrow \mathbf{P}^3 \times \mathbf{P}^3$ under β. We claim that

$\mathcal{X} := \alpha^{-1}(\Sigma) \cup \beta^{-1}(\Delta)$ is a flat family over $\overline{\mathcal{H}''}$, and this will guarantee a morphism from $\mathrm{Bl}_\Sigma(\mathcal{H}_{2,0,3} \times \mathbf{P}^3) \longrightarrow \mathcal{H}_{2,-1,3}$ by the universal property of the Hilbert scheme.

$$\begin{array}{ccc} \mathcal{X} := \alpha^{-1}(\Sigma) \cup \beta^{-1}(\Delta) & \overset{i}{\hookrightarrow} & \overline{\mathcal{H}''} \times \mathbf{P}^3 \\ & \searrow^{\phi} & \downarrow \\ & & \overline{\mathcal{H}''}(:= \mathrm{Bl}_\Sigma(\mathcal{H}_{2,0,3}) \times \mathbf{P}^3) \end{array} \tag{3.6}$$

To show flatness, we employ the following proposition.

Proposition 3.5.6. *A family $\mathcal{X} \subset \mathbf{P}^r_B$ of closed subschemes of a projective space over a reduced connected base B is flat if and only if all fibers have the same Hilbert polynomials.*

Reference. [5, Proposition III-56]. □

Lemma 3.5.7. *$\phi : \mathcal{X} \longrightarrow \overline{\mathcal{H}''}$ is a flat family.*

Proof. The family is clearly flat away from Σ. Now within Σ, the family is still flat over points which represent singular or smooth conics with embedded points on the smooth components of the curves; one way to see this is to check the Hilbert polynomial and see that it agrees with $2m + 2$ (as we do with other cases), but another way is to simply observe that locally analytically there is an open set which corresponds to an open set of $\mathcal{H}_{1,-1,3}$ since a smooth component of a curve is locally analytically isomorphic to a line.

Lastly we would like to check that the family is flat over the points which represent singular conics with embedded points at the node or double lines with embedded points. Since $\overline{\mathcal{H}''}$ is reduced, by Proposition 3.5.6 it is enough to check that the fibers all have the same Hilbert polynomials. In Section 4.2, we show that these curves belong to finite orbits under the action of PGL_4. Namely, there are six different orbits for these cases. Since projectively equivalent curves have the same Hilbert polynomials, it suffices to check that each representative of these six orbits has the Hilbert polynomial equal to $2m + 2$. Appendix B exhibits the Macaulay2 calculations for the Hilbert polynomials of these six representatives, which are indeed all $2m + 2$. The proof is complete. □

Lemma 3.5.8. *There exists a bijective morphism $f : \overline{H''} \longrightarrow \mathcal{H}''$.*

Proof. First of all, from Lemma 3.5.7, we have a morphism f from $\overline{\mathcal{H}''}$ to $\mathcal{H}$ by the universal property of the Hilbert scheme. The map is actually to $\mathcal{H}$, but by construction, it invariably maps to the ghost component only. Let us analyze what this map is doing pointwise.

Away from the exceptional divisor, f maps $([C], p)$ to the corresponding point in $\mathcal{H}'' \subset \mathcal{H}$. Given a point on the Σ, the fiber is a projectivization of the normal bundle, which is isomorphic to $\mathbf{P}^1$. A point on the exceptional divisor corresponds to a nonzero section of the normal bundle, so for the fibers over a point which represents a conic and an embedded point on the smooth component, we map a point over the fiber to the point on $\mathcal{H}''$ parametrizing $([C], p)$ such that p is embedded in C with the specified normal direction. For a point on the fiber of the point corresponding to a singular conic and an embedded point at the node, we are mapping it the following way. Say we have $I_C = (xy, z)$. Then the blow-up at the origin is $\mathrm{Proj}(k[x, y, z])[A, B]/(Az - Bxy)$. Then we can map each point on the fiber to the corresponding ideal

$$(x^2y, xy^2, zx, zy, z^2, Az - Bxy),$$

which represents a singular conic and an embedded point.

The fact that the map is injective here is a consequence of Example 2.1.12 and Example 2.2.8 where we showed that there is a unique representation of these embedded points given the coordinate system. So the morphism f is an injection. But the image $f(\overline{\mathcal{H}''})$ is a closed subvariety of $\mathcal{H}$ and contains the general conic and a point. Therefore it must contain the closure of this component; this shows that f is surjection. (Refer to Figure 4.2.) Hence f is a bijective morphism. This proves Theorem 3.5.1.c). □

Remark 3.5.9. There is an alternative way to prove Lemma 3.5.8. Instead of exhibiting a flat family over $\mathcal{H}''$ we could have just checked that the corresponding set-theoretic map described above is equivariant under change of coordinates (this involves some work), and then prove the continuity of the map. Then by Corollary 2.4.9, we can extend the map to a morphism.

We end the section with a conjecture, which, if true, would tell us the exact shape of $\mathcal{H} = \mathcal{H}' \cup \mathcal{H}''$.

Conjecture 3.5.10. *Both $\mathcal{H}'$ and $\mathcal{H}''$ are smooth and are isomorphic to $\overline{\mathcal{H}'}$ and $\overline{\mathcal{H}''}$ respectively.*

Suggested method of proof. Given the smoothness of these components, isomorphisms follow from Zariski's Main Theorem since $\overline{\mathcal{H}'}$ and $\overline{\mathcal{H}''}$ are both smooth and we have bijective morphisms. While it is not known that these components are smooth, there is a good reason to believe this conjecture to be true. In [28], Piene and Schlessinger[3] showed that the two components of $\mathcal{H}_{3,0,3}$ are both smooth and that they intersect transversally. Their technique involved looking at two types of curves to one of which every other curve (parametrized by $\mathcal{H}_{3,0,3}$) specialized, and by constructing the universal deformation space of these two curves, they successfully proved Theorem 3.8.1 listed at the end of this chapter. It seems plausible that the same technique could be applied in this case. The two cases we have to check are: 1) a double line without embedded points (such as $I_C = (x^2, xy, y^2, zy - x)$, here we can carry out a method similar to that of Example 2.1.12), and 2) a double line with a planar embedded point (such as $I_C = (x^2, xy, y^2, zy)$). In the latter case, one needs to calculate the deformation space over two different components, $\mathcal{H}'$ and $\mathcal{H}''$. □

3.6 $\mathcal{H}_{2,-n,3}$

We are not going to be dealing too much with other negative genuses, but there are a few things one can say about $\mathcal{H}_{2,-n,3}$. The first question is, which subschemes of $\mathbf{P}^3$ are parametrized by points of $\mathcal{H}_{2,-n,3}$? Of course, the easy answer is: these are precisely the subschemes which have $2m + n + 1$ as the Hilbert polynomial. The reduced scheme should be of degree 2 or 1. By a similar argument as in Section 3.5, we conclude that the reduced component of the subscheme should be a plane conic or a pair of skew lines. Clearly the scheme as a whole would contain additional points (either embedded or separate) most of the time. And these subschemes can be inductively determined by $\mathcal{H}_{2,-(n-1),3}$. Another way of saying this is that, the ghost component should consist of points which parametrize a pair of a point and another subscheme which has $2m + (n-1) + 1$ as the Hilbert polynomial; $\mathcal{H}_{2,-(n-1),3}$ already contains all such subschemes.

In addition to the ghost component, we have an irreducible component in $\mathcal{H}_{2,-n,3}$ whose general points parametrize double lines of genus $-n$ without any embedded points. Specifying a double line L in $\mathbf{P}^3$, as we saw, amounts to specifying a normal direction to L, or equivalently to specifying a map ϕ from L to $\mathbf{P}^3/L$. (To clarify, this is a vector space quotient. Another way to write this would be $\mathbf{C}^2 \longrightarrow \mathbf{C}^4/\mathbf{C}^2 \cong \mathbf{C}^2$.) The degree of ϕ can be any nonnegative numbers. In the genus -1 case, we restricted our attention to only the linear maps. In Section 2.3, we saw that the degree of this map is the negative of the arithmetic genus of the line. Consequently, we have the following theorem.

Theorem 3.6.1. *$\mathcal{H}_{2,-n,3}$ contains an irreducible component whose points parametrize double lines of genus $-n$. The dimension of this component is $5 + 2n$.*

Proof. The dimension of this component is determined by choosing the line (that's 4), and specifying two degree n homogeneous polynomials in two variables up to scaling, which is $2n + 1$. □

3.7 $\mathcal{H}_{3,1,3}$

We now study the space of plane cubics. By Hartshorne's classification of curves in $\mathbf{P}^3$ [16, IV. 6], plane cubics have genus 1, and so their Hilbert polynomials are $P_C(m) = 3m$.

[3]The author had the pleasure to communicate with both Ragni Piene and Michael Schlessinger. They confirmed that the smoothness of these components is not known, but they both agreed that the same technique as in [28] should work.

Theorem 3.7.1. *$\mathcal{H}_{3,1,3}$ parametrizes plane cubics and their specializations in $\mathbf{P}^3$. The dimension is 12, and it is isomorphic to the $\mathbf{P}^9$-bundle $\mathrm{PSym}^3\mathcal{S}^*$ where $S \longrightarrow \mathbf{P}^{3*}$ is the universal subbundle.*

Lemma 3.7.2. *If $C \in \mathbf{P}^3$ has Hilbert polynomials $3m$, then it is a plane cubic or its specialization.*

Proof. Our strategy is to push for an exhaustive method. While this method does work here, there are clearly a lot of different cases to consider. We start with C_{red} once again, and $\deg(C_{\text{red}}) \le 3$.

i) Suppose C_{red} is contained in a plane. If $\deg(C_{\text{red}}) = 3$ and C_{red} is irreducible, C_{red} is a plane cubic having $3m$ as the Hilbert polynomial. So $C_{\text{red}} = C$.

ii) If C_{red} is planar, $\deg(C_{\text{red}}) = 3$ and C_{red} is reducible, then it is either a union of a conic and a line (necessarily intersecting since we are in $\mathbf{P}^2$) or union of three intersecting lines. In each of these cases, we see by Section 4.3 that $P_{C_{\text{red}}}(m) = 3m$ and $C_{\text{red}} = C$. Both cases are specializations of plane cubics.

iii) If C_{red} is planar, $\deg(C_{\text{red}}) = 2$, and irreducible, then C_{red} is a plane conic. But then we get a contradiction. Since a conic is a degree 2 Veronesian embedding of $\mathbf{P}^1$, the Hilbert polynomial would necessarily have an even number as the coefficient of the leading term, which $3m$ does not.

iv) If C_{red} planar, $\deg(C_{\text{red}}) = 2$, and reducible, we have two lines. In $\mathbf{P}^2$ and two distinct lines intersect, so C_{red} is a singular conic. Since $\deg C = 3$, one of the lines must have the multiplicity two structure. So we know that C contains at least the intersection of a line and a double line. So we write $C = {C_{\text{red}}}' \cup Z$ where ${C_{\text{red}}}'$ is the union of a line and a double line, and Z is the union of all zero-dimensional subschemes of C. We claim ${C_{\text{red}}}'$ is planar; this equivalent to saying that the double line is planar (one of genus 0) and the single line lies in that plane. Since ${C_{\text{red}}}'$ is connected, if it is not planar, we can proceed as in step *vii)* and we arrive at the conclusion that ${C_{\text{red}}}'$ is contained in a smooth or singular quadric surface. Then $P_{{C_{\text{red}}}'}(m) = 3m + 1$ or $3m + 3$, both of which are too big. So ${C_{\text{red}}}'$ must indeed be planar. But then this is a degeneration of a plane cubic, which has the Hilbert polynomial $3m$ already. Therefore Z must be empty and $C = {C_{\text{red}}}'$ is a specialization of a plane cubic curve.

v) If C_{red} is planar and $\deg(C_{\text{red}}) = 1$, then C_{red} is a line and by the classification of triple lines in Section 2.3 (specifically Remark 2.3.4) we see that this triple line is a specialization of a plane cubic.

vi) If C_{red} is non-planar, then C_{red} cannot be a twisted cubic or any of its degenerations (in Section 4.4) since it would have the Hilbert polynomial $3m + 1$ which is already too big.

vii) If C_{red} is non-planar and is *connected*, then by utilizing the sequence (where H is a hyperplane)

$$0 \longrightarrow \mathcal{O}(D-1) \longrightarrow \mathcal{O}(D) \longrightarrow \mathcal{O}(D)|_H \longrightarrow 0$$

we have the following:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}) & \longrightarrow & H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1)) & \longrightarrow & H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}) & \longrightarrow & H^0(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}(1)) & \longrightarrow & H^0(\Gamma, \mathcal{O}_{\mathbf{P}}(1)) & \longrightarrow & 0 \end{array} \tag{3.7}$$

Since H is a general hyperplane Γ is just three points, so $h^0(\Gamma, \mathcal{O}_{\mathbf{P}}(1)) = 3$. Since C_{red} is connected, $h^0(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}) = 1$. In turn we have $h^0(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}(1)) \le 4 = h^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$. If we look at the same sequence twisted once,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1)) & \longrightarrow & H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2)) & \longrightarrow & H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}(1)) & \longrightarrow & H^0(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}(2)) & \longrightarrow & H^0(\Gamma, \mathcal{O}_{\mathbf{P}}(2)) & \longrightarrow & 0 \end{array} \tag{3.8}$$

Now since

$$h^0(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}(2)) \le h^0(\Gamma, \mathcal{O}_{\mathbf{P}}(2)) + h^0(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}(1)) \le 3 + 4 = 7,$$

and $h^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2)) = 10$, we have a family of quadric surfaces containing C_{red}, and this has dimension at least $3 = 10 - 7$, and this cannot all be quadric surfaces of rank 2 or 1. We reason this out as follows: by assumption C_{red} is not planar, so it cannot lie on a rank 1 quadric surface; if it lies on a 3-dimensional family of quadric surfaces and they were all rank 2, by intersecting 2 or more of them we arrive at a conclusion that it is planar, which is a contradiction. In other words, the curve has to lie on either a smooth quadric or a singular quadric.

If C lies on a smooth quadric, it is either type $(0, 3)$ or $(1, 2)$ (all other cases being equivalent to one of these). In the first case the Hilbert polynomial is $3m + 3$ and in the second case the Hilbert polynomial is $3m + 1$. So this case does not occur.

If the curve lines on a cone, then it must pass through the vertex odd number of times because the degree is odd. If it passes through the vertex 3 times, then the curve reduces down to a single point and this is ridiculous. So we conclude the curve passes through the origin exactly once, and meets a general line on the conic one more time. If we blow up the cone along the vertex, the Picard group of the blow up is generated by the exceptional divisor E and a general line on the cone L, and we have $C \cdot E = 1$ and $C \cdot L = 2$. But then this is linearly equivalent to a twisted cubic, whose Hilbert polynomial is $3m + 1$. So this, too, does not occur.

viii) If C_{red} is non-planar, and it is not connected, and is of degree 3, then we have either a conic and a line or three lines. If C_{red} is a conic and a line, nonplanar, they must intersect otherwise $P_{C_{\text{red}}}(m) = (2m + 1) + (m + 1) = 3m + 2 > 3m$. But since C_{red} is nonplanar, they can only intersect transversely at one point, and this is a degeneration of a twisted cubic curve as shown in Section 4.4. Then the Hilbert polynomial of C_{red} is already $3m + 1$. This case does not occur.

ix) If C_{red} is non-planar, and it is not connected, and is of degree 3 such that we have three lines. The only case where C_{red} is not a degeneration of a twisted cubic curve is if we have a singular conic and a disjoint line, or three disjoint lines. For the first case the Hilbert polynomial of C_{red} is already $(2m+1)+(m+1) = 3m+2$, which is too large. Likewise for the latter we have $3m + 3$, which is also too big. This case does not occur.

x) If C_{red} is nonplanar, not connected, of degree 2, we must have two skew lines. Then C must contain at least a double line and a single line (and possibly other zero-dimensional components). We saw in Section 2.3 that the Hilbert polynomial of a double line is at least $2m + 1$. And the Hilbert polynomial of a line is at least $m + 1$. So this case does not occur.

We check that there are no other cases since there are no nonplanar irreducible conics and since all lines are planar. The proof is complete. □

The rest of the proof is analogous to the case with $\mathcal{H}_{2,0,3}$. Since there are no ghost components, we can construct a map

$$\mathcal{H}_{3,1,3} \longrightarrow \mathbf{P}^{3*}$$

and proceed to conclude

$$\mathcal{H}_{3,1,3} \cong \mathbf{P}(\mathrm{Sym}^3 \mathcal{S}^*)$$

where $\mathcal{S} \longrightarrow \mathbf{P}^3$ is the universal hyperplane subbundle. Towards the end of the proof, we use Theorem 3.1.4 just like the other case. We can also check the smoothness of $\mathcal{H}_{3,1,3}$ by Example 2.1.7:

$$\begin{aligned} \dim T_C \mathcal{H}_{3,1,3} &= h^0(C, \mathcal{N}_{C/\mathbf{P}^3}) = h^0(C, \mathcal{O}_C(3) \oplus \mathcal{O}_C(1)) = h^0(C, \mathcal{O}_C(3)) + h^0(C, \mathcal{O}_C(1)) \\ &= 9 + 3 = 12 = \dim \mathbf{P}(\mathrm{Sym}^3 \mathcal{S}^*) = \dim_{[C]} \mathcal{H}_{3,1,3}. \end{aligned}$$

From this Zariski Main Theorem applies, and the map is an isomorphism.

Remark 3.7.3. This theorem naturally lets us question whether or not the result holds true in general – that is, is it always the case that the Hilbert scheme parametrizing plane curves of degree d in $\mathbf{P}^3$ are always of the form $\mathbf{P}(\mathrm{Sym}^d(\mathcal{S}^*))$? The answer turns out to be yes. Again, the second half of each argument stands analogous to that of $\mathcal{H}_{2,0,3}$. In addition, we already know from [16, V] that plane curves of degree d have genus

$$g = \frac{(d-1)(d-2)}{2}.$$

The task, thus, boils down to showing that a one-dimensional subscheme having the Hilbert polynomial equal to

$$dm - \frac{(d-1)(d-2)}{2} + 1$$

is necessarily planar. The author has tried various ways to prove this assertion. If anything, the length and the complexity of the proof of Lemma 3.7.2 tell us that this is, in general, *not* the way to approach this problem. The author would like to include in this section a step towards proving this assertion.

To do this, we define the following operation. Given a subscheme C in P^3, let $S(C)$ = the maximal subscheme of C which is locally Cohen-Macaulay, which is of pure dimension 1. How do we know such a thing exist? There are two ways to construct $S(C)$. One way is to take the closure of the union of all the points of C at which the local ring is Cohen-Macaulay. Another way, more algebraic, to get the same thing is as follows: We look at the primary decomposition of the ideal defining C. While the primary decomposition is not unique, the ideals whose associated primes are *minimal* are uniquely determined. (And the rest are indeed the embedded primes.) We take J to be the ideal constructed by taking the intersection of all primary ideals of the primary decomposition of I whose associated primes are minimal. Then J is uniquely determined, and indeed the zero ideal (0) of R/J determines such a maximal subscheme. Since localization commutes with everything, we can glue together this construction to get $S(C)$. These two constructions in fact agree.

Geometrically, what is going on? We think of this as a way to remove all the embedded points of C while keeping all multiplicity. What do we know about the operation S?

- $S(S(C)) = S(C)$
- The Hilbert polynomial of $S(C)$ is less than or equal to the Hilbert polynomial of C.

Let $P_C(m)$ denote the Hilbert polynomial of C. So suppose we have a curve C in $\mathbf{P}^3$. If C is not planar, we can always choose a plane onto which to project C such that we get an additional spatial embedded point *as we take the flat limit of a one-parameter family.* This is not possible for planar curves because we get a whole component projected onto another component, instead of two local lines which intersect transversally. Let C' be the projection of C constructed by means of taking the flat limit. Then $P_C(m) = P_{C'}(m)$ since it's a flat limit. Meanwhile $P_{C'}(m)$ strictly greater than $P_{S(C')}(m)$ since C' contains at least one spatial embedded point. So $P_C(m) < P_{S(C')}(m)$.

Now suppose $S(C')$ is a planar curve. Then $S(C')$ is a planar curve of degree d, which would have genus $(d-1)(d-2)/2$. So

$$P_C(m) > P_{S(C')}(m) = dm - g + 1.$$

And C is not parametrized by Hilb^{dm-g+1}.

This argument takes care of all the cases where $S(C')$ is planar. These include all reduced curves and curves whose projection is planar (double lines which are *ribbons*, for instance, become planar double lines with an embedded point under the flat transformation. Check Example 2.2.7). The curves which do not fall under this category include *ropes*. An example of one such curve is a triple line given as a square of an ideal defining a line, e.g. (x^2, xy, y^2).

It is not clear exactly how to attack this case. Karen Chandler has kindly pointed out that we have the following fact regarding *ropes*: in $\mathbf{P}^3$, suppose we have a smooth and reduced curve C of degree d and genus g defined by the ideal I_C. Then the subscheme defined by the ideal $(I_C)^2$ has genus

$$5(g-1) + 4d + 1.$$

We may be able to use this fact to complete this argument.

Yet another way to approach this problem is to generalize Castelnuovo's bound whose corollary asserts that for smooth curves, plane curves achieve the highest genus, and this is precisely the expected genus. This argument entails looking at hyperplane sections and realizing that the minimal Hilbert polynomial of the hyperplane sections is achieved when the d points lie in a line *as a scheme.* Other than that, there are no other clear ways to deduce this fact.

3.8 $\mathcal{H}_{3,0,3}$

Cubic curves in $\mathbf{P}^3$ with genus 0 are truly interesting since the most general type of these are irreducible, reduced, nonplanar and nondegenerate curves. These are known as the *twisted cubics*. What do we know about the space of twisted cubic curves? We can calculate its dimension in the following manner. We know that all twisted cubic curves are projectively equivalent. We can simply fix one such curve $C_0 \in \mathcal{H}_{3,0,3}$. We can construct a set map from $\mathrm{PGL}_4 \longrightarrow \mathcal{H}_{3,0,3}$ where $g \mapsto g(C_0)$. The dimension of the space of all transformations which fix C_0 is 3 since it is a map from $\mathbf{P}^1$ to $\mathbf{P}^1$ (in other words, $g \in \mathrm{PGL}_2$). So we conclude that the dimension of the space of the twisted cubic curves is

$$\dim \mathrm{PGL}_4 - \dim \mathrm{PGL}_2 = 15 - 3 = 12. \tag{3.9}$$

This is good since the dimension of the Hilbert scheme at a point corresponding to a twisted cubic is 12 by Theorem 3.1.4.

A priori, however, we know that there is an additional component of the Hilbert scheme, namely the ghost component representing a plane cubic and a point. (We will use the same notation as in Section 3.5 – that is, $\mathcal{H}'$ will denote the component whose points parametrize twisted cubic curves, and $\mathcal{H}''$ will denote the ghost component.) This component's dimension is necessarily 15 (the space of plane cubics has dimension 12 as we saw in Section 3.7, and the space of points is 3-dimensional). By Example 2.1.7, if the point is separate from the plane cubic, we also know that

$$h^0(C, \mathcal{N}_{C/\mathbf{P}^3}) = h^0(C_{\mathrm{red}}, \mathcal{O}_C(3) \oplus \mathcal{O}_C(1)) + 3 = 12 + 3 = 15. \tag{3.10}$$

We can say a little bit about this Hilbert scheme. For instance, we know by Example 2.1.12 that the tangent space at this point is 12-dimensional, in accordance with the dimension we calculated above in 3.9. Hence we have just proved that at a generic point this component is smooth. So the Hilbert scheme is smooth at the point $[C]$ corresponding to a triple line (without any embedded points). But we actually proved a bit more than this. Notice in Figure 4.4 (at the end of Chapter 4) that a curve in the orbit (XVII) is obtained as a flat limit of those curves in (XIV). But a flat limit of nonreduced curves cannot become reduced. So $\mathcal{H}$ has to be smooth at $[C']$ where $C' \in$ XIV. Inductively, we conclude that the Hilbert scheme is smooth *along every orbit containing C in its closure.* This is the compliment of $\mathcal{H}' \cap \mathcal{H}''$ in $\mathcal{H}''$. By the same line of reasoning and looking at 3.10 we can say that $\mathcal{H}''$ is smooth at every point $[C]$ where $[C]$ is a plane cubic and a *separate* point. But when the point is embedded within the curve, the situation is far from clear.

Once we establish that these are the only components of the Hilbert scheme, we know what the intersection of these two components should contain – a curve with an embedded point that is a degeneration of a general twisted cubic curve. A nodal cubic curve with a spatial embedded point is the most general type. Piene and Schlessinger have done some remarkable work using the techniques of deformation theory to give a detailed description of $\mathcal{H}_{3,0,3}$. In this section we merely state the rest of the results.

Theorem 3.8.1 (Piene & Schlessinger). *1. $\mathcal{H}_{3,0,3}$ consists of two irreducible components, $\mathcal{H}'_{3,0,3}$ of dimension 12 and $\mathcal{H}''_{3,0,3}$ of dimension 15. Both $\mathcal{H}'$ and $\mathcal{H}''$ are smooth and rational, they intersect transversally, and their intersection is nonsingular, rational, of dimension 11.*

2. If $C \in \mathcal{H}' \cap \mathcal{H}''$, then C is a plane, singular cubic curve with a spatial embedded point, "emerging from" the plane, at a singular point. More precisely, C is projectively equivalent to the curve defined by an ideal $I \in k[x, y, z, w]$ of the form $I = (xz, yz, z^2, q(x, y, w))$ where $q(x, y, w)$ is a cubic form which is singular at $(x, y, w) = (0, 0, 1)$.

3. If $[C] \in \mathcal{H}' \cap \mathcal{H}''$, then $\dim T_{[C]}\mathcal{H}_{3,0,3} = 16$.

4. The scheme $\mathcal{H}'$ decomposes as a finite disjoint union of affine spaces, $\mathcal{H}' = \mathbf{A}^{12} \cup U\mathbf{A}^{n_i}$, where $0 \leq n_i \leq 11$ and all integers between 0 and 11 occur.

Reference. [28, pp. 761–773]. □

3.9 $\mathcal{H}_{3,-n,3}$

We end this chapter by including comments about some recent developments that have been made in regards to $\mathcal{H}_{3,-n,3}$. As of yet, the complete descriptions of $\mathcal{H}_{3,-n,3}$ are not available. But people have looked at the non-ghost component. This is the component that parametrizes curves which are locally Cohen-Macaulay. We denote this by $H(d,g)$. For instance Martin-Deschamps and Perrin [20] showed that $H(3,-1)$ is smooth and irreducible of dimension 8 (1996). In general, however, $H(3,g)$ for $g < -1$ is not irreducible. Nollet [25] showed that $H(3,-2)$ contains two components: one component representing the union of a double line of genus -3 and a line meeting it, and the other representing three disjoint lines (1997). The latter is of dimension 12. By analyzing the ideal sheaves of the extremal curves to which all other curves of $H(3,g)$ specialize (and their subsequent resolutions), Nollet successfully calculated the dimensions of each component that is included in $H(3,g)$ for $g < -1$. In addition, he showed that $H(3,g)$ is connected for all genus. (Although the connectedness of $\mathcal{H}_{3,-n,3}$ is a special case of Hartshorne's theorem [17], it is not known that both components, that is, the ghost-component and the non-ghost component, are individually connected.) Beyond this, our understanding of the complete Hilbert schemes (including the ghost components) of degree three curves are limited.

Chapter 4

Strata of Curves in $\mathbf{P}^3$

We go back to an important notion we came across in Chapter 3. As we mentioned before the group PGL_4 can act on the coordinates of $\mathbf{P}^3$ by matrix multiplication. These correspond to invertible linear transformations. What do we know about these transformations? Clearly two distinct points cannot be mapped onto one, and information regarding colinearity and coplanarity is preserved.

We will be using the following definitions for the rest of this chapter. We say two curves are *projectively equivalent* if one can be obtained as an image of the other under the action of PGL_4. An *orbit* is set of all curves which are projectively equivalent; it is a projective equivalence class. We define a *stratum* (pl. *strata*) to be a component of $\mathcal{H}_{d,g,r}$ which consists of points parametrizing all the curves in an orbit. We say an orbit B is a *specialization* of the orbit A if there exists a one-parameter family of curves in A whose limit is a curve in B. In this case, we will also say that a curve C in the orbit B is a *specialization* (or a *flat limit*) of the curves in the orbit A. Finally a *stratification diagram* is a diagram indicating the dimension of each stratum as well as the inclusion relationships under closure.

Notice that in general a stratum is not a closed subset of $\mathcal{H}_{d,g,r}$ since its closure should contain all of the curves of the corresponding orbit *as well as* all of their specializations. Instead, it is only "locally closed," i.e. it is an open subset of a closed set. In contrast, if a certain orbit has no more specializations, then the corresponding stratum is a closed subset. Additionally it is a closed subvariety since it is the image of an irreducible variety.

Therefore, one way to visualize a Hilbert scheme and its distinct components is to look at the strata of the parametrized curves. In some cases there are finitely many orbits under the action of PGL_4, and hence finitely many strata in $\mathcal{H}_{d,g,r}$.

In this chapter, we exhibit the complete strata of curves which are parametrized by the points of the Hilbert schemes we encountered in Chapter 3. We will go beyond simply exhibiting the strata by calculating the dimension of each orbit (which, in turn, would correspond to the dimension of the stratum of the Hilbert scheme), and showing how an element of one stratum can be achieved as a flat specialization of curves from another stratum. In doing this, we are correcting and completing the diagrams that appear in Chapter 1 of [13]. This relationship will tell us that one stratum of $\mathcal{H}_{d,g,r}$ lies in in the closure of another. To exhibit a flat family, it suffices to give a parametrization in terms of t and check the continuity of the dimension; this corresponds to choosing an arc on the $\mathcal{H}_{d,g,r}$. In addition we will be able to determine the curves which are parametrized by both the ghost component and the irreducible component of the Hilbert schemes.

It is clear that all lines in $\mathbf{P}^3$ are projectively equivalent. In addition, for $\mathcal{H}_{1,-1,3}$ we already saw that there are two orbits – one with a separate point and one with an embedded point. This chapter will start with $\mathcal{H}_{2,0,3}$. In general $\mathcal{S}_{d,g,r}$ will refer to the stratification diagram of curves which are parametrized by the points of $\mathcal{H}_{d,g,r}$. And when necessary, we will be using $\mathcal{S}'$ and $\mathcal{S}''$ to refer to the components of the stratification diagram corresponding to the irreducible component and the ghost component, respectively. When we are only interested in the local geometry (which is most of the time), we will be using the affine coordinates to give examples of specializations; nevertheless, there will be situations where a flat specialization can only be realized through projective coordinates (See Section 4.4, (II)).

Figure 4.1: $\mathcal{S}_{2,0,3}$

4.1 $\mathcal{S}_{2,0,3}$

We have already seen two facts:

i) a general point here corresponds to a plane conic;
ii) there are no ghost components in $\mathcal{H}_{2,0,3}$.

So there are total of 3 strata in $\mathcal{H}_{2,0,3}$.

(I) *A nonsingular plane conic.* This is of dimension 8. We specify the plane (that's 3), and the family of conics on the plane is 5. This is the most general type.

(II) *Singular conics.* This is a specialization of (I). Consider the one parameter family of smooth conics: $\{x^2 - y^2 - t = 0\}_{t\in\mathbf{C}}$. As $t \longrightarrow 0$ the curve will become singular at the origin. To calculate the dimension, we just choose the plane (that's 3), a point as the node (that's 2), and two directions (that's 2). The dimension is 7.

(III). *Planar double lines.* This is a specialization of (II). Consider the family of singular conics: $\{x^2 - (ty)^2 = 0\}_{t\in\mathbf{C}}$. When $t \longrightarrow 0$, we get a double line defined by the ideal (x^2). Here we choose the plane and a line on the plane. That's 5.

Remark 4.1.1. Figure 4.1 gives a diagrammatic representation of $\mathcal{S}_{2,0,3}$ with connected lines indicating specializations. Since the lower dimensional space will always be the specialized one, there is no ambiguity. Even though it is possible for a smooth conic to specialize to double lines without passing through the stage of being singular, since we are only interested in the inclusion relationship, we do not need to connect (I) and (III) separately.

Remark 4.1.2. We will use the term *singularize* to describe the behavior of the flat family specialization going from (I) to (II). And *singularization* is just any one parameter family which exhibits a similar kind of behavior. We will encounter more complex diagrams as the degree increases or as an extra embedded point gets introduced. From now on, when we deal with a component of a subscheme which is a smooth plane conic, we can just say *singularize the conic* without any ambiguity. Likewise we will describe the flat family going from (II) to (III) as *closing the singular conic.*

4.2 $\mathcal{S}_{2,-1,3}$

We know a priori that this stratification diagram should consist of two distinct (not disjoint, however) parts since there is a ghost component. Furthermore these two parts should intersect since the Hilbert scheme should be connected. Since the ghost component deals with an extra point, we need to separate the cases where the point lies on the plane defined by the conic and where the point lies off that plane. These clearly belong to two different orbits. First of all, we have the following sets of subschemes:

There are four strata of curves belonging to $\mathcal{H}'_{2,-1,3}$, the most general of them being two skew lines.

(I) *Two skew lines.* Since $G(1,3)$ is the space of lines in $\mathbf{P}^3$ and has dimension 4, the space of two lines in $\mathbf{P}^3$ has dimension 8.

(II) *A singular plane conic with a spatial embedded point.* If two skew lines approach to form a pair of incident lines, the flat limit will have an embedded point. This is a specialization of (I). Example 2.2.4 illustrates this scenario. To calculate the dimension of this space, this is same as the space of singular conics (since the embedded point is determined to be at the node once we choose a singular conic.) And we saw in the previous section that this is of dimension 7. We will see that there is another component which specializes to (II).

(III) *A double line without embedded points.* We choose the line (that's 4). Now assigning a normal direction at every point is equivalent to choosing a linear map from $\mathbf{P}^1 \longrightarrow \mathbf{P}^3/\mathbf{P}^1 \cong \mathbf{P}^1$ where the quotient is taken as vector spaces; therefore, the space of maps has dimension 3 (since it equals $\dim \mathrm{PGL}_2$). So the dimension is 7. This is clearly a specialization of (I). Example 2.2.6 illustrates this case.

(IV) *A double line with a spatial embedded point.* That this is a specialization of (II) is easy to see. We apply the same technique we used in Section 4.1 (closing the singular conic) without the embedded point. To see that this is also a specialization of (III), refer to Example 2.2.7.

The dimension is then easy to calculate. We can choose a line as the support (that's 4), and choose a plane on which the line lies (that's 1), and choose a point to be embedded (that's 1). The rest is specified including all other normal directions. The dimension is 6. We will later see that another component specializes to (IV).

In addition, we have the ghost component $\mathcal{H}_{2,-1,3}$ whose general member is a smooth plane conic C and a point. There are 16 strata in the ghost component, two of which we have already encountered.

(V) *A smooth plane conic C with a point $p \notin C$.* Since the space of conics on a given plane has dimension 5, we just need to choose the plane (that's 3) and a point (also 3). So the total dimension is 11.

(VI) *A smooth plane conic C with an embedded point p.* Notice that as the point approaches the conic, the flat limit will be a conic with an embedded point. So this is a specialization of (V). Here we choose the conic (that's 8 as seen above), and a point on the conic (that's 1) and the normal direction (that's 1). The dimension is 10.

The one-parameter family examined in Example 2.2.3 best exemplifies this case, even though there the connected subscheme is a line. What is important is the fact that in $\mathbf{P}^3$ (unlike in $\mathbf{P}^2$) the flat limit preserves some information about the direction of the approach.

(VII) *A singular plane conic C with a point $p \notin C$.* This is a specialization of (V) by singularizing the conic. Here we choose the plane (that's 3), a point and two directions (that's 4) and a point (that's 3). The dimension is 10.

(VIII) *A singular plane conic C with an embedded point $p \in C - C_{sing}$.* This is a specialization of (VI) by singularizing the conic. This is also a specialization of (VII) by Example 2.2.3. We choose the plane and the two lines like before (that's 7 total), and we need to choose a point from one line or another (that's 1), and a normal direction (that's 1). The dimension is 9.

(IX) *A planar double line and a point $p \notin C$.* This is a specialization (VII) by closing the singular conic. The dimension is 8. We already know the dimension of planar double lines is 5, and we choose a point in $\mathbf{P}^3$.

[For the spaces (V) through (IX), we can consider further specializations by requiring the point p to lie on the plane on which the curve C lies. We will denote them by (V'), (VI'), (VII'), (VIII'), and (IX'). The dimension of each of these clones is one less than its original space.]

(X) *A singular plane conic C with an embedded point of general type at the node of C.* Such a curve occurs as a specialization of (VIII). This is clear. Here we choose the singular plane curve (that's 7), and determine the point $[a, b] = [\lambda_0, \lambda_1] \in \mathbf{P}^1$ (that's 1). The dimension is 8. Example 2.2.8 describes this scenario. Notice also that this *specializes* to (II) by sending $a \longrightarrow 0$ in Example 2.2.8.

(XI) *A singular plane conic C with a planar embedded point at the node.* This is clearly a specialization of (X) by sending $b \longrightarrow 0$ in Example 2.2.8. Notice also that this could be obtained by sending the embedded point on the plane from $C - C_{sing}$ to the node. Therefore it also specializes (VIII'). The dimension in this case is just 7 since once the singular conic is chosen, the embedded point type is specified.

(XII) *A double line C with an embedded point of general type.* This is a planar double line except with an embedded point. So here we choose the line (that's 4), the plane containing the line (that's 1), the point (that's 1) and the point $[a, b] \in \mathbf{P}^1$ (that's 1). The dimension is 7. This is a specialization of (X) by closing the singular conic. It also specializes (IX) in ways similar to Example 2.2.8. In addition, this specializes to (IV) by letting $a \longrightarrow 0$.

(XIII) *A double line C with a planar embedded point.* This is clearly a specialization of (XI) by closing the singular conic and of (XII) by letting the plane approach the double line along the plane. Here we choose the line, the plane, and the point (that's 4+1+1). The dimension is 6.

Combining all this we have the above strata (Figure 4.2). For instance, the intersection of the ghost component and the irreducible component (the shaded portion) has points which parametrize singular conics with spatial embedded points. Its closure also contains the points which represent doubles lines with spatial embedded points.

4.3 $\mathcal{S}_{3,1,3}$

Drawing all of the strata of $\mathcal{H}_{3,0,3}$ is impossible because among nonsingular cubics there are infinitely many projective equivalence classes. While we know that all the nonsingular plane cubics in $\mathbf{P}^2$ can be written in Weierstrass normal forms [8, 15.2]:

$$Y^2 = 4X^3 - \alpha XZ^2 - \beta Z^3,$$

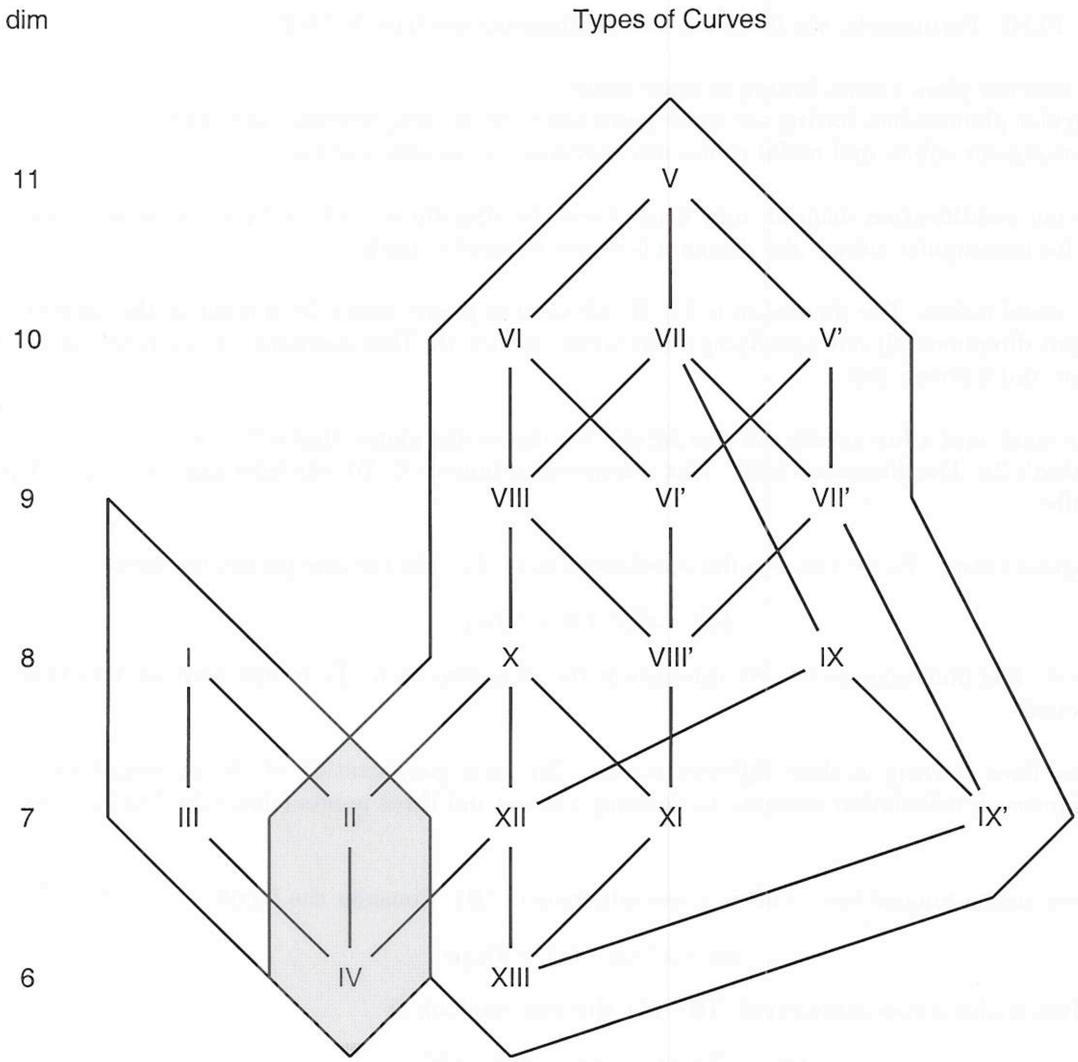

Figure 4.2: $\mathcal{S}_{2,-1,3}$

where α and β are constants, they are not projectively equivalent for certain different (α, β)'s. Another way to write the equation defining smooth cubics is the following:

$$Y^2 Z = X(X - Z)(X - \lambda Z).$$

And two such curves C_λ and $C_{\lambda'}$ are equivalent if and only if the j-invariants

$$j(\lambda) = 256 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 \cdot (\lambda - 1)^2}$$

coincide [12, 10.16]. Fortunately, the following three statements are true [8, 15.2]:

i) All the singular plane cubics belong to finite orbits;
ii) All singular plane cubics having the same geometric type are projectively equivalent;
iii) All nonsingular cubics and nodal cubics can specialize to cuspidal cubics.

Therefore our stratification diagram may begin from the stratum of nodal cubics. Once we ignore the components for nonsingular cubics, the diagram becomes relatively simple.

(I). *Plane nodal cubics.* The dimension is 11. We choose the plane (that's 3), a point as the node (that's 2), two tangent directions (2) and specifying cubic terms (that's 4). This specializes to cuspidal cubics and also to a conic and a secant line.

(II). *Plane conic and a line meeting at two points.* We choose the plane (that's 3), a conic (that's 5), and two points (that's 2). The dimension is 10. This is a specialization of (I). We can take $\{xy - x^3 + ty^3 = 0\}_{t \in \mathbf{C}}$ as a flat family.

(III). *Cuspidal cubics.* To see this is a flat specialization of (I), take the one parameter family

$$\{y^2 - x^2(x + t) = 0\}_{t \in \mathbf{C}}$$

and let $t \longrightarrow 0$. The dimension is 10. We calculate it the same way as in (I), except here we have only one tangent direction.

(IV). *Three lines meeting at three different points.* This is a specialization of (II) by singularizing the conic. The dimension calculation amounts to choosing a plane and three points (that's 3+2+2+2), and it is 9.

(V). *A conic and a tangent line.* This is a specialization of (II). Consider the family

$$\{(y - x^2)(y - tx) = 0\}_{t \in \mathbf{C}}$$

as $t \longrightarrow 0$. This is also a specialization of (III). For this one, we look at

$$\{y^2 - x^3 + (1 - t)(x^3 - yx^2) = 0\}_{t \in \mathbf{C}}$$

which is cuspidal for $t \neq 0$, but then at $t = 0$ the curve becomes the union of a conic (defined by $y - x^2 = 0$) and a line $(y - x = 0)$. Here we choose the plane, the conic and a point (that's 3+5+1). The dimension is 9.

(VI). *Three concurrent lines.* This is a specialization of (IV). We can consider

$$\{(x - y)(x + y)(x + t) = 0\}_{t \in \mathbf{C}}$$

as $t \longrightarrow 0$. This is also a specialization of (V) by singularizing the conic. We choose the plane, the point and three directions. The dimension is 8.

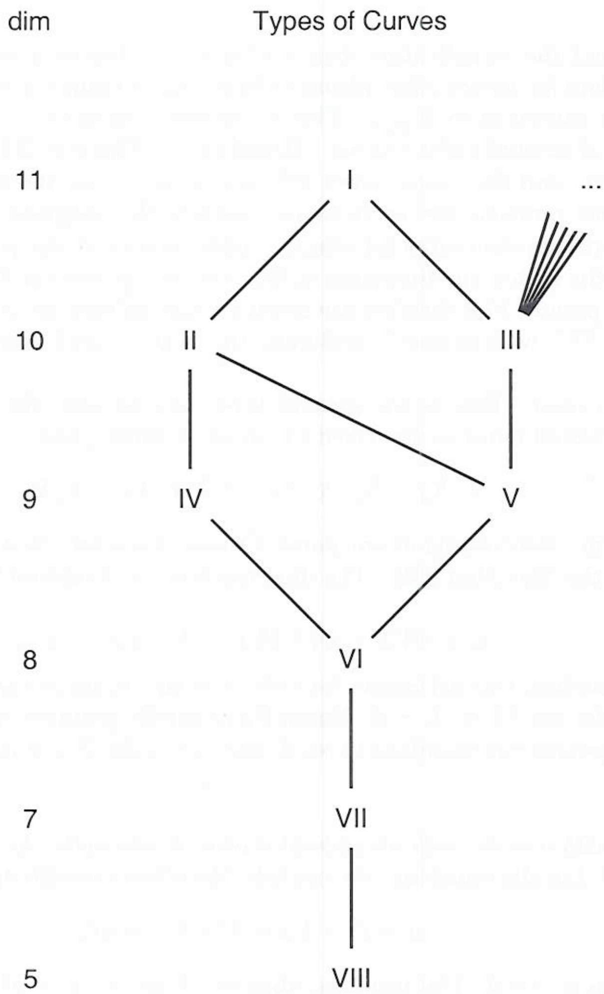

Figure 4.3: $\mathcal{S}_{3,1,3}$ without the Nonsingular Planar Cubics

(VII). *A planar double line and a single line meeting at one point.* This is a specialization of (VI) by closing a singular conic from (VI). The dimension is 7.

(VIII). *Triple line.* Clearly this is a specialization of (VII). The dimension is 5, just choosing one line and a plane that contains the line.

The stratification diagram is exhibited in Figure 4.3. The lines emerging to (III) represent the infinitely many orbits of smooth plane cubics specializing to (III).

4.4 $\mathcal{S}'_{3,0,3}$

We could try to construct the stratification diagram for $\mathcal{H}_{3,0,3}$, but we run into the same problem, namely the infinitude of the orbits for nonsingular planar cubics. But it turns out that there are only finitely many orbits if we restrict out attention to $\mathcal{H}'_{3,0,3}$. This is the component that whose points parametrize curves that are degenerations of twisted cubic curves. Therefore, by Theorem 3.8.1.2 all embedded points in this diagram are spatial ones, and the planar ones will not occur in the intersection of $\mathcal{H}'$ and $\mathcal{H}''$. We will be relying heavily on our previous two sections to complete this diagram. In particular, for the strata of curves whose reduced scheme structures are singular plane conics, if the position of the embedded point is predetermined (e.g. at the node) the dimension of the orbit is the same as the space of singular plane conics without any embedded points. Flat families are given for specializations that are not obvious. For the ones with the asterix mark ('**'), we will also be including the MACAULAY2 program verification in Appendix C.

(I). *Smooth twisted cubics.* This is the general type. As we saw, the dimension is 12. In projective coordinates, a typical twisted curve is described by an ideal looking like

$$(X_0X_2 - X_1^2, X_1X_3 - X_2^2, X_0X_3 - X_1X_2).$$

(II). *A conic and a line, intersecting at one point.* Choose the conic (that's 8), a point on the conic (that's 1), and a direction for the line (that's 2). The dimension is 11. Consider the following family of ideals in projective coordinates:

$$I_t = (WZ - tXY, WY - X^2, XZ - tY^2).$$

For $t = 1$ this is the standard twisted cubic; for t=0, it is the union of the conic defined by $WY = X^2$ in the plane $Z = 0$ with the line $W = X = 0$. Ragni Piene kindly pointed out that geometrically, this is the family obtained by projecting the standard twisted cubic onto the $Z = 0$ plane from the point $(0, 0, 0, 1)$ on the cubic.[1]

(III). *A plane cubic with a node, with an embedded point at the node.* As in 4.3, the dimension is 11. This is a specialization of (I). Locally speaking, we can take the affine coordinates

$$x = t^2 - 1, y = t^3 - 1, z = at,$$

and consider the limit as $a \longrightarrow 0$. The resulting ideal is $(z^2, yz, xz, y^2 - x^2(x + 1))$, which is a nodal cubic curve with a spatial embedded point.

(IV). *Three lines, non-coplanar.* Choose a pair of skew lines (that's 8) and a point from each (that's 2). The dimension is 10. This is a specialization of (II) by singularizing the conic.

(V). *A conic and a line meeting at 2 points, with an embedded point at one.* As in 4.3, this is of dimension 10, and is a specialization of (III). This is also a specialization of (II) (just fold the line onto the plane containing the conic, and we saw in Example 2.2.4 how a spatial embedded point arises) by the same argument as in Example 2.2.4.

(VI). *A plane cubic with a cusp, with an embedded point at the cusp.* As in 4.3, the dimension is 10. This is a specialization of (III). Take

$$I = (z^2, yz, xz, y^2 - x^2(x + t))_{t \in \mathbf{C}}$$

for instance. Then the limiting ideal as $t \longrightarrow 0$ becomes $(z^2, yz, xz, y^2 - x^3)$. This is a cusp with an embedded point.

(VII). *A double line of genus -1, and a line meeting it (and lying in its projective tangent space at the point of intersection).* The space of double lines of genus -1 in $\mathbf{P}^3$ is 7 as we saw. Then we just need to

[1] For the case of *complete twisted cubics* Ragni Piene [27] exhibited all 11 Schubert degenerations of twisted cubics curves by similar means.

choose a point and a direction for the other line (that's 2). The dimension is 9. This is a specialization of (IV).** For instance, take the set of lines defined by

$$V(tx + (1-t)y, z-t) \cup V(y, z) \cup V(y, x).$$

It can be checked easily that as t goes to 0 is $(zy, xz+y) = (y^2, zy, xz+y)$. At $x \neq 0$ we have a usual double line of genus -1. At $x = 0$ we have the z-axis and *there is no embedded point at the origin.*

(VIII). *Three concurrent, non-coplanar lines.* Choose a singular conic (that's 7) and an extra direction off the plane (that's 2). The dimension is 9. This is a specialization of (IV). Here take the set of lines defined by

$$V(x, y) \cup V(x, z-t) \cup V(y, z).$$

It can be checked easily that as $t \longrightarrow 0$, the limiting ideal becomes (xy, yz, zx), which defines three axes. That is, the spatial embedded point that usually occurs when two skew lines become incident is no longer, as we are intersecting it with (x, y).

(IX). *Three coplanar lines, with an embedded point at one point of intersection.* As in 4.3, the dimension is 9 and is a specialization of (V). This could also be achieved by bringing together two skew lines in (IV) but not along the third line. We get an embedded point at the intersection as in Example 2.2.4.

(X). *A conic and a tangent line, with an embedded point at their intersection.* As in 4.3, the dimension is 9, and it is a specialization of (VI) and (V).

(XI). *Planar double line (of genus 0) with a line* not *lying on its plane.* Choose a line and a plane containing it (that's 4+1=5), and a point on the line (that's 1) and a direction off the plane (that's 2). The dimension is 8. That this is a specialization of (VIII) comes from Example 2.2.5 where we take $N_t = (y, z-tx)$ instead of $(y-x, z-tx)$. This time there is no embedded point and the ideal becomes (yx, yz, z^2).** We can also look at it as closing a singular conic. This is also a specialization of (VII). Take $(y^2, zy, xz+y)$ from (VII), and consider the transformation $z \mapsto tz$ as $t \longrightarrow \infty$. Alternatively we can think of this as $I((sy)^2, z(sy), xz+sy)$ and let $s \longrightarrow 0$. The limiting ideal is (y^2, zy, zx).

(XII). *Planar double line and a line in its plane with an embedded point somewhere on the double line.* Here the embedded point is not predetermined. Choose a line and a plane containing it (that's 4+1=5), a point on the line and a direction within the plane (that's 1+1=2), and a point on the double line to be embedded (that's 1). The dimension is 8. This is a specialization of (X) by singularizing the conic. Also, it is a specialization of (IX) by closing the singular conic containing the embedded point. This is also a specialization of (VII) by embedding a point in the double line by means of Example 2.2.7.

(XIII). *Three concurrent and coplanar lines, with an embedded point at their common point.* As in 4.3, the dimension is 8. By Example 2.2.5, this is a specialization of (VIII). To see this as a specialization of (IX), take $V(z^2, xz, xy, yz) \cup V(z, y+x-t)$ and let $t \longrightarrow 0$.** The limiting ideal is

$$(z^2, xz, yz, xy(x+y)),$$

which agrees with the form of Example 2.2.5. This is also a specialization of (X) by singularizing the conic.

(XIV). *Triple line lying on a quadric cone – i.e., first order normal direction constant, second order normal varying.* The dimension is 7. Here we notice that there are 8 dimensional family of cones in $\mathbf{P}^3$, and given a cone we choose the line on it (that's 1). But we also note that given such a line, there are two-dimensional family of quadric cones that contain the line (in general, given a twisted cubic curve, there are two dimensional family of quadric surfaces that contain it). So the dimension is $8+1-2=7$. This is a specialization of (XI). Take (y, z^2) as the double line. We will write this as $(y, xy-z^2)$ to remind ourselves that it lies on the cone defined by $xy = z^2$. And we consider the line parametrized by $(y - tz, z - tx)$. It's clear that this lies on the same cone. Now the ideal of the union of these two components is

$$(xy - z^2, y(y - tz), y(z - tx)). **$$

Let t approach 0, and the ideal is of the form $(xy - z^2, y^2, yz)$.

Remark 4.4.1. The interesting thing about this line (XIV) is that at every point except one the Zariski tangent space has dimension 2, and at one point (which corresponded to the vertex of the cone) the dimension is 3. This is to be expected since the dimension of the Zariski tangent space is upper-semicontinuous; however there are no embedded points since there are no embedded primes.

(XV). *Planar double line and a line in its plane, with an embedded point at their intersection.* Choose a line and a plane containing it (that's 5), and a point and a direction within the plane (that's 1+1=2). The dimension is 7. It's fairly obvious that (XV) is a specialization of (XIII) (by closing a pair of lines together), of (XII) (by one parameter family which moves the line without embedded point towards the point), and of (XI) (by rotating the line not on the plane onto the plane). In the last case, that the embedded point pops up can be checked again by using Example 2.2.5, but using $L \cup M$ as (z, y^2).

(XVI). *Planar triple line, with an embedded point.* The dimension is 6 since it's determined by the flag of line-point-normal direction. This is a specialization of (XV) by rotating the single line towards the double line (closing the conic in a sense). This is also a specialization of (XIV). Consider $(xz - y^2, yz, z^2)$ and consider the change of variable $z \mapsto tz$ and let $t \longrightarrow \infty$. Equivalently, we could send $y \mapsto sy$ and have s approach 0. The resulting ideal is (xz, yz, z^2).

(XVII). *Triple line given by the square of the ideal of a line.* Choose a line in $\mathbf{P}^3$ and you're done. The dimension is 4. This is a specialization of (XIV). Take $(xz - y^2, yz, z^2)$ and take the change of variable $y \mapsto ty$ and let $t \longrightarrow \infty$. Likewise we could take $x \mapsto sx$ and let s go to 0. The resulting ideal is $(y^2, yz, z^2) = (y, z)^2$.

Figure 4.4 on the next page depicts $\mathcal{S}'_{3,0,3}$. The shaded portion corresponds to $\mathcal{S}'_{3,0,3} \cap \mathcal{S}''_{3,0,3}$ all of whose strata have spatial embedded points. Notice that any stratum in this area cannot specialize to $\mathcal{S}'_{3,0,3} - \mathcal{S}''_{3,0,3}$ – that is, you cannot *un*embed a point through flat limits.

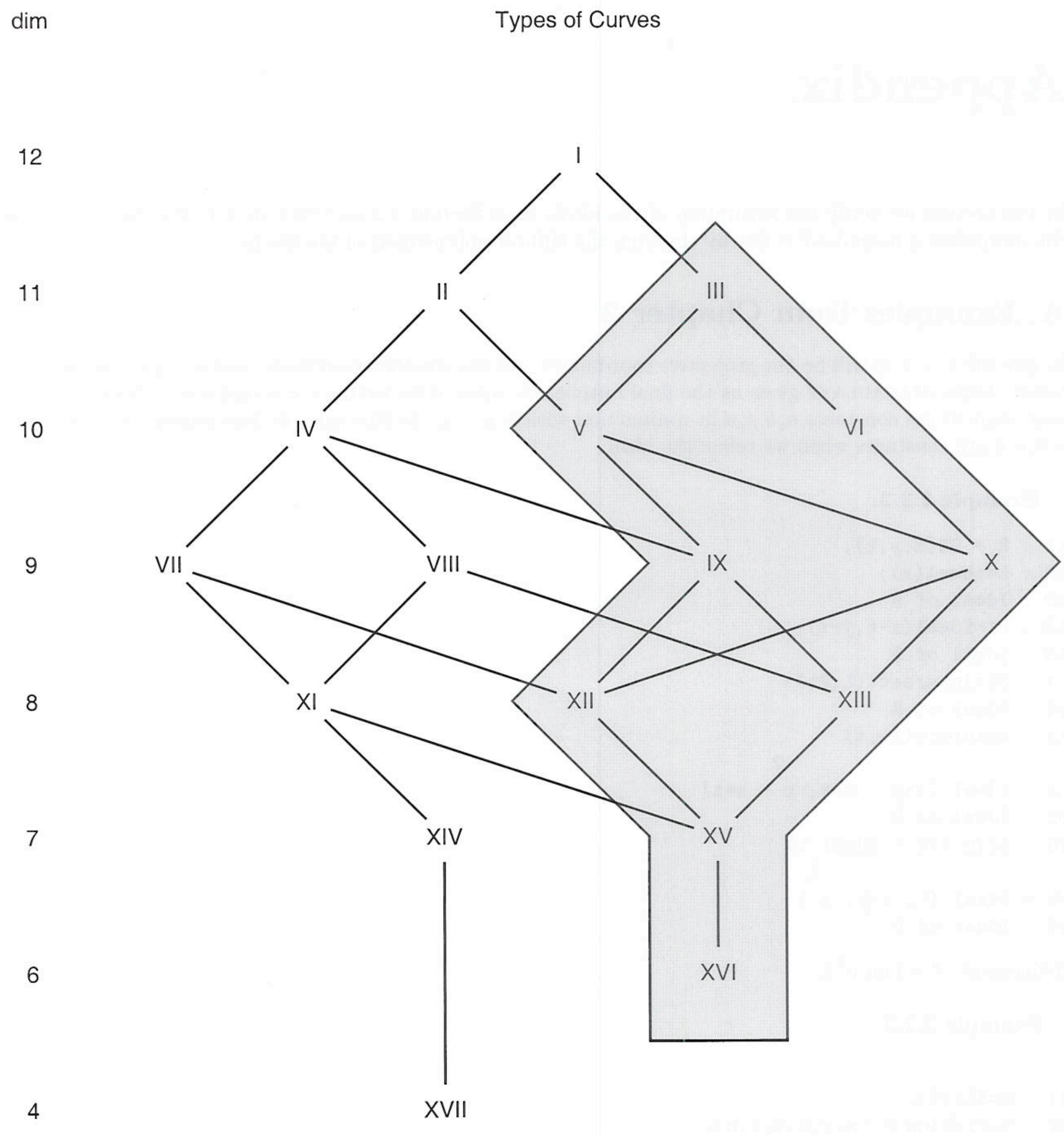

Figure 4.4: $\mathcal{S}'_{3,0,3}$

Appendix

In this section we verify the saturation of the ideals from Section 2.2 and Section 4.4. We also prove that the morphism ϕ from 3.5.7 is flat by checking the Hilbert polynomials of the fibers.

A. Examples from Chapter 2

In general, x, y, z, w will be our projective coordinates. t is the variable coordinate, and s is the homogeneous factor. Once MACAULAY2 gives us the final output, we *refine* it by letting $t = 0$ and $s = 1$. Sometimes we may want to use constants a, b, c, d in conjunction with x, y, z, w. In this case, we just remind ourselves that a, b, c, d are constants when we refine the ideal.

Example 2.2.2.

```
i1 : R = QQ[x,y,t];
i2 : L=ideal(x);
o2 : Ideal of R
i3 : Pt=ideal(x-t,y-t);
o3 : ideal of R
i4 : Xt=intersect(L,Pt);
o4 : Ideal of R
i5 : saturate(Xt,t)
                              2
o5 = ideal (x*y - x*t, x - x*t)
o5 : Ideal of R
i6 : trim (Xt + ideal t)
                         2
o6 = ideal (t, x*y, x )
o6 : Ideal of R
```

Refinement: $I = (xy, x^2)$.

Example 2.2.3.

```
i1 : k=ZZ/101;
i2 : variables = (x,y,z,w,t,s);
i3 : constants = (a,b,c);
i4 : R=k[variables,constants];
i5 : L=ideal(x,z);
o5 : Ideal of R
i6 : Pt = ideal(x*s-a*t,y*s-b*t,z*s-c*t);
o6 : Ideal of R
i7 : Xt=intersect(L,Pt);
```

```
o7 : Ideal of R
i8 : saturate(Xt, t);
o8 : Ideal of R
i9 : trim(Xt + ideal t)

                                         2                         2
o9 = ideal (t, z*s*a - x*s*c, z s, y*z*s, x*z*s, x*y*s, x s)
o9 : Ideal of R
```

Refinement: $I = (az - cx, z^2, xz, yz, xy, x^2)$.

Example 2.2.4.

```
i1 : R = QQ[x,y,z,w,t,s];
i2 : Lt = ideal(y, z-t);
o2 : Ideal of R
i3 : M = ideal(x, z);
o3 : Ideal of R
i4 : Xt = intersect(Lt, M);
o4 : Ideal of R
i5 : saturate(Xt, t)
                2
o5 = ideal(z  - z*t, y*z, x*z - x*t, x*y)
o5 : Ideal of R
i6 : trim(Xt+ideal t)
                 2
o6 = ideal (t, z , y*z, x*z, x*y)
o6 : Ideal of R
```

Refinement: $I = (z^2, yz, xz, xy)$.

Example 2.2.5.

```
i1 : R = QQ[x,y,z,w,t,s];
i2 : Xt = ideal(z*(z*s-t*x),z*(z*s-t*y),(z*s-t*x)*(z*s-t*y));
o2 : Ideal of R
i3 : saturate(Xt, t)
                                         2                  2    2
o3 = ideal(x*z - y*z, y*z*t - z s, x*y*t - y*z*s, x y - x*y )
o3 : Ideal of R
i4 : trim(Xt + ideal t)
                                     2          2       2
o4 = ideal (t, x*z - y*z, z s, y*z*s, x y - x*y )
o4 : Ideal of R
```

Refinement: $I = (xz, z^2, yz, xy(x - y))$.

Example 2.2.6. We omit this example since the answer depends on the variable functions.

Example 2.2.7. We omit this example since it does not require saturation.

Example 2.2.8.

```
i1 : R = QQ[x,y,z,w,a,b,t,s];
i2 : C = ideal(x*y,z);
o2 : Ideal of R
```

```
i3 : Pt = ideal(x-t, y-t, a*z*s-b*t^2);
o3 : Ideal of R
i4 : Xt = intersect(C, Pt);
o4 : Ideal of R
i5 : saturate(Xt, t)
                                                     2          2
o5 = ideal (y*z - z*t, x*z - z*t, x*y*b - z*a*s, x*y  - x*y*t, x y - x*y*t)
o5 : Ideal of R
i6 : trim (Xt + ideal t)
                                             2   2
o6 = ideal (t, y*z, x*z, x*y*b - z*a*s, x*y , x y)
o6 : Ideal of R
```

Refinement: $I = (yz, xz, az - bxy, xy^2, x^2y)$ and the term z^2 could be included also, but it would be redundant.

B. Constancy of the Hilbert Polynomials of the Fibers under $\phi : \mathcal{X} \longrightarrow \overline{\mathcal{H}''}$

Here we include the proof that the Hilbert polynomial of each fiber is indeed $2m + 2$, completing the proof of Lemma 3.5.6 that $\phi : \mathcal{X} \longrightarrow \overline{\mathcal{H}''}$ is a flat family. Because PGL_4 acts on the space, we only need to check a representative of each class. We assume the family is locally flat over all points which represent either 1) a plane conic and a separate point, and 2) a plane conic and an embedded point on the smooth component of the conic. The author is indebted to Professor Daniel Grayson from University of Illinois at Urbana-Champaign for providing the following codes.

```
i21 : k = ZZ/101;
i22 : variables = (x,y,z,w); -- coordinates of the variable point
i23 : constants = (
      F,G,            -- blow-up direction
      X,Y,Z,W,        -- coordinates of the parameter point
      a,b,c,          -- coefficients of the equation of the plane
      p,q,r,s,t,u     -- coefficients of the eqn. of the conic on that plane
      );
i24 : R = k[variables, constants];
```

Now we construct the homogeneous coordinate ring of the blow-up using the two equations of the incidence correspondence.

```
i25 : A = R/( F*(p*X^2+q*X*Y+r*Y^2+s*Y*Z+t*X*Z+u*Z^2) -
              G*(a*X+b*Y+c*Z+W) )
o25 = A
o25 : QuotientRing
```

Here are the two equations that cut out the family of conic curves.

```
i26 : f=a*x+b*y+c*z+w;
i27 : g=p*x^2+q*x*y+r*y^2+s*y*z+t*x*z+u*z^2;
i28 : I = ideal(f,g);
```

And here are the equations that cut out the family of points. 'minors' is a useful way to cut out the diagonal.

```
i29 : J = minors(2, matrix{{x,X},{y,Y},{z,Z},{w,W}})
o29 = ideal (- y*X + x*Y, - z*X + x*Z, - z*Y + y*Z, - w*X + x*W, - w*Y + y*W,
 - w*Z + z*W)
o29 : Ideal of A
```

Now we take the union of the two families

```
i30 : K = intersect(I, J);
o30 : Ideal of A
```

We then make a ring for $\mathbf{P}^3$ and specialize.

```
i31 : S = k[x,y,z,w];
```

Case 1. Singular conic & embedded point at the node. (Assume $G = 0$.)

```
i32 : ev = o-> substitute(o, matrix {{
      x,y,z,w,         -- leave this
      1,0,             -- F,G : one of these must be nonzero
      0,0,1,0,         -- X,Y,Z,W : one of these must be nonzero
      0,0,0,           -- a,b,c
      0,1,0,0,0,0      -- p,q,r,s,t,u : one of these must be nonzero
      }})
o32 = ev
o32 : Function
```

Our choice of coordinates represents a singular conic (z, xy), and we chose the origin as our point, which is the node. We have to check the fiber over a given point when $G = 0$, F nonzero, and when $F = 0$, and G nonzero, and when they are both nonzero. The above case is when $G = 0$.

The following line guarantees that the values we choose satisfy the equation of the blow-up!

```
i33 :  assert(0==(
         (ev F)*(ev (p*X^2+q*X*Y+r*Y^2+s*Y*Z+t*X*Z+u*Z^2))
       - (ev G)*(ev (a*X+b*Y+c*Z+W)
      )))
```

We compute the Hilbert polynomial now.

```
i34 : K' = trim ev K
                2
o34 = ideal (w , y*w, x*w, x*y)
o34 : Ideal of S
i35 : hilbertPolynomial (S/K')
o35 = 2*P
         1
o35 : Projective HilbertPolynomial
```

The output o35 means the Hilbert Polynomial is twice that of $\mathbf{P}^1$, which we know to be $m + 1$. So the Hilbert polynomial of this fiber is 2m+2, which is what we wanted.

Case 2. Singular conic & embedded point at the node when $F = 0$.

```
i36 : ev = o-> substitute(o, matrix{{x,y,z,w, 0,1,   0,0,1,0,  0,0,0,
                                          0,1,0,0,0,0 }});
i37 : K' = trim ev K
              2                     2   2
o37 = ideal (w , z*w, y*w, x*w, x*y , x y)
o37 : Ideal of S
i38 : hilbertPolynomial (S/K')
o38 = 2*P
         1
o38 : ProjectiveHilbertPolynomial
```

Case 3. Singular conic & embedded point at the node when $F, G \neq 0$.

```
i39 : ev = o->substitute(o,matrix{{x,y,z,w,  1,1,  0,0,1,0,  0,0,0,
                                         0,1,0,0,0,0  }});
i40 : K' = trim ev K
              2
o40 = ideal (w , y*w, x*w, x*y - z*w)
o40 : Ideal of S
i41 : hilbertPolynomial (S/K')
o41 = 2*P
         1
o41 : ProjectiveHilbertPolynomial
```

Case 4. Double line & embedded point when $F = 0$.

```
i42 : ev = o -> substitute(o,matrix {{x,y,z,w,  0,1,  0,0,1,0,   0,0,0,
                                 1,0,0,0,0,0 }});
i43 : K' = trim ev K
              2                  2   3
o43 = ideal (w , z*w, y*w, x*w, x y, x )
o43 : Ideal of S
i44 : hilbertPolynomial (S/K')
o44 = 2*P
         1
o44 : ProjectiveHilbertPolynomial
```

Case 5. Double line & embedded point when $G = 0$.

```
i45 : ev = o->substitute(o,matrix{{x,y,z,w,  1,0,  0,0,1,0,  0,0,0,
                                          1,0,0,0,0,0  }});
i46 : K' = trim ev K
              2            2
o46 = ideal (w , y*w, x*w, x )
o46 : Ideal of S
i47 : hilbertPolynomial (S/K')
o47 = 2*P
         1
o47 : ProjectiveHilbertPolynomial
```

Case 6. Double line & embedded point when $F, G \neq 0$.

```
i48 : ev = o-> substitute(o,matrix{{x,y,z,w,  1,1,  0,0,1,0,  0,0,0,
                                  1,0,0,0,0,0 }});
```

```
i49 : K' = trim ev K
               2             2
o49 = ideal (w , y*w, x*w, x  - z*w)
o49 : Ideal of S
i50 : hilbertPolynomial (S/K')
o50 = 2*P
         1
o50 : ProjectiveHilbertPolynomial
```

These exhaust all the cases. Of course, one could go ahead and check the other cases where the embedded point is on a smooth component of the curve; and when we do we will likewise verify that the Hilbert polynomial is $2m + 2$.

C. Saturation of the Select Ideals from Chapter 4

This section includes some of the less obvious verifications from Section 4.3.

(IV) $\longrightarrow$ (VII).

```
i7 : R = QQ[x,y,z,w,t,s];
i8 : A = ideal(t*x+(s-t)*y, z-t);
o8 : Ideal of R
i9 : B = ideal(y, z);
o9 : Ideal of R
i10 : C = ideal(x, y);
o10 : Ideal of R
i11 : D = intersect(A, B, C);
o11 : Ideal of R
i12 : saturate(D, t)
o12 = ideal (y*z - y*t, x*z - y*t + y*s)
o12 : Ideal of R
i13 : trim(D + ideal t)
o13 = ideal (t, y*z, x*z + y*s)
o13 : Ideal of R
```

Refinement: $I = (yz, xz + y) = (y^2, zy, xz + y)$.

(VIII) $\longrightarrow$ (XI).

```
i13 : R = QQ[x,y,z,w,t,s];
i14 : A = ideal(y, z);
o14 : Ideal of R
i15 : B = ideal(x, z);
o15 : Ideal of R
i16 : Ct = ideal(y, z*s - t*x);
o16 : Ideal of R
i17 : Dt = intersect(A, B, Ct);
o17 : Ideal of R
i18 : saturate(Dt, t)
                                2
o18 = ideal (y*z, x*y, x*z*t - z s)
o18 : Ideal of R
i19 : trim(Dt + ideal t)
```

```
                           2
o19 = ideal (t, y*z, x*y, z s)
o19 : Ideal of R
```

Refinement: $I = (yz, xy, z^2)$.

(IX) $\longrightarrow$ (XIII).

```
i19 : R = QQ[x,y,z,w,t,s];
i20 : A = ideal(z^2,x*z,x*y,y*z);
o20 : Ideal of R
i21 : Bt = ideal(z,y+x-t);
o21 : Ideal of R
i22 : Ct = intersect(A, Bt);
o22 : Ideal of R
i23 : saturate(Ct, t)
                2              2       2
o23 = ideal (z , y*z, x*z, x y + x*y  - x*y*t)
o23 : Ideal of R
i24 : trim(Ct + ideal t)
                   2               2       2
o24 = ideal (t, z , y*z, x*z, x y + x*y )
o24 : Ideal of R
```

Refinement: $I = (z^2, yz, xz, xy(x + y))$.

(XI) $\longrightarrow$ (XIV).

```
i26 : R = QQ[x,y,z,w,t,s];
i27 : A = ideal(y, x*y - z^2);
o27 : Ideal of R
i28 : Bt = ideal(y*s - t*z, z*s - t*x);
o28 : Ideal of R
i29 : Ct = intersect(A, Bt);
o29 : Ideal of R
i30 : saturate(Ct, t)
                     2              2   2
o30 = ideal (x*y - z , y*z*t - y s, z t - y*z*s)
o30 : Ideal of R
i31 : trim (Ct + ideal t)
                        2           2
o31 = ideal (t, x*y - z , y*z*s, y s)
o31 : Ideal of R
```

Refinement: $I = (xy - z^2, y^2, yz)$.

Bibliography

[1] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of Algebraic Curves, vol. 1*, New York, Springer-Verlag, 1985.

[2] Ciro Ciliberto and Edoardo Sernesi, Families of varieties and the Hilbert scheme, *Lectures on Riemann surfaces (Trieste, 1987), pp. 428–499*, Teaneck, World Sci. Publishing, 1989.

[3] David Cox, John Little and Donald O'Shea, *Ideals, Varieties, and Algorithms*, New York, Springer-Verlag, 1996.

[4] David Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, New York, Springer-Verlag, 1995.

[5] David Eisenbud and Joseph Harris, *The Geometry of Schemes*, New York, Springer-Verlag, 2000.

[6] David Eisenbud and A. Van de Ven, On the Normal Bundles of Smooth Rational Space Curves, *Mathematische Annalen, vol. 256, pp. 453–463*, New York, Springer Verlag, 1981.

[7] William Fulton, *Intersection Theory*, New York, Springer Verlag, 1984.

[8] C. G. Gibson, *Elementary Geometry of Algebraic Curves*, Cambridge, Cambridge University Press, 1998.

[9] Joshua C. Greene, *The Existence of Hilbert Schemes*, Unpublished senior thesis, Harvard University, 1997.

[10] Alexandre Grothendieck, Techniques de construction et théorém d'existence en géométrie algébrique IV: Les schémas de Hilbert, *Séminaire Bourbaki, vol. 6, no. 221, pp. 249–276*, Paris, Soc. Math. France, 1995.

[11] Phillip Griffiths and Joseph Harris, *Principles of Algebraic Geometry*, New York, John Wiley & Sons, 1978.

[12] Joseph Harris, *Algebraic Geometry: A First Course*, New York, Springer-Verlag, 1995.

[13] Joseph Harris, *Curves in Projective Space*, Montréal, Les Presses de L'Université de Montreal, 1982.

[14] Joseph Harris and Ian Morrison, *Moduli of Curves*, New York, Springer-Verlag, 1991.

[15] Joseph Harris and Klaus Hulek, On the Normal Bundle of Curves on Complete Intersection Surfaces, *Mathematische Annalen, vol. 264, pp.129–135*, New York, Springer-Verlag, 1983.

[16] Robin Hartshorne, *Algebraic Geometry*, New York, Springer-Verlag, 1977.

[17] Robin Hartshorne, Connectedness of the Hilbert Scheme, *Institut des Hautes Études Scientifiques, vol. 29, pp. 5–49*, Paris, Publication Mathématique, 1966.

[18] János Kóllar, *Rational Curves on Algebraic Varieties*, New York, Springer-Verlag, 1996.

[19] O. A. Laudal, Formal Moduli of Algebraic Structures, *Lecture Notes in Mathematics, vol. 754*, New York, Springer-Verlag, 1979.

[20] M. Martin-Deschamps and D. Perrin, Le Schéma de Hilbert des Courbes Gauches localement Cohen-Macaulay n'est (presque) jamais réduit, *Annales Scientifiques de l'École Normale Supérieure (4), vol. 29, pp. 757–785*, Paris, Gauthier-Villars, 1996.

[21] Hideyoki Matsumura, *Commutative Ring Theory,* Cambridge, Cambridge University Press, 1986.

[22] David Mumford, *Abelian Varieties,* Oxford, Oxford University Press, 1974.

[23] David Mumford, *Lectures on Curves on an Algebraic Surface*, Studies in Mathematics, vol. 59, Princeton, Princeton University Press, 1968.

[24] David Mumford, The Red Book of Varieties and Schemes, *Lecture Notes in Mathematics, vol. 1358*, New York, Springer-Verlag, 1988.

[25] Scott Nollet, The Hilbert Schemes of Degree Three Curves, *Annales Scientifiques de l'École Normale Supérieure (4), vol. 30, no. 3, pp. 367–384*, Paris, Gauthier-Villars, 1997.

[26] Christian Okonek, Michael Schneider and Heinz Spindler, *Vector Bundles on Complex Projective Spaces*, Boston, Birkhauser Boston, 1980.

[27] Ragni Piene, Degenerations of Complete Twisted Cubics, *Enumerative Geometry and Classical Algebraic Geometry, Progress in Mathematics, vol. 24, pp. 37–50*, Boston, Birkhauser, 1982.

[28] Ragni Piene and Michael Schlessinger, On the Hilbert Scheme Compactification of the Space of Twisted Cubics, *American Journal of Mathematics, vol. 107, no. 4, pp. 761–774*, Baltimore, Johns Hopkins University Press, 1985.

[29] Ragni Piene, On the Problem of Enumerating Twisted Cubics, *Lecture Notes in Mathematics, vol. 1124, pp.329–337*, New York, Springer-Verlag, 1983.

[30] Igor R. Shafarevich, *Basic Algebraic Geometry 1: Varieties in Projective Space*, New York, Springer-Verlag, 1988.

[31] Igor R. Shafarevich, *Basic Algebraic Geometry 2: Schemes and Complex Manifolds*, New York, Springer-Verlag, 1988.

[32] Stein A. Strømme, Elementary Introduction to Representable Functors and Hilbert Schemes, *Parameter Spaces,* Warszawa, Banach Center Publications, 1996.