Homework 1 - Number Theory

Exercise 1. For each of the following algebraic numbers α,

$$\frac{1+\sqrt{8}}{2}, \cos(\frac{2\pi}{5}), \exp(\frac{\pi i}{3}),$$

find the minimal polynomial of α over $\mathbb{Q}$ and show that it is irreducible in $\mathbb{Q}[X]$.

Which of these numbers are algebraic integers? For each α which is not an algebraic integer, find the smallest natural number N such that $N\alpha$ is an algebraic integer.

Exercise 2. This exercise shows there is an infinite number of non-isomorphic quadratic number fields. (In contrast, there are only 7 non-isomorphic quadratic extensions of the 2-adic numbers $\mathbb{Q}_2$.)

a) If F is a quadratic number field, show $F = \mathbb{Q}(\sqrt{m})$ for some square-free integer m.

b) Show uniqueness of m in **a**) in the following precise sense: If m and m' are square-free integers not equal to 0 or 1 and $\phi\colon \mathbb{Q}(\sqrt{m}) \to \mathbb{Q}(\sqrt{m'})$ is a field isomorphism, then $m = m'$.

Exercise 3. The only integer solution of the equation

$$X^3 = Y^2 + 1$$

is given by $X = 1$ and $Y = 0$.

Exercise 4. a) Determine the units in $\mathbb{Z}[\zeta_3]$ where ζ_3 is a primitive third root of unity.

b) Let $p > 3$ be a prime number and ζ_p is a primitive pth root of unity. Show that $1 + \zeta_p$ is a unit in $\mathbb{Z}[\zeta_p]$. Is it a root of unity?

Exercise 5. a) Let $\overline{\mathbb{Z}}_M^d$ be the set of algebraic integers α of degree at most d over $\mathbb{Q}$ such that all conjugates of α have absolute value bounded by M. Show that $\#\overline{\mathbb{Z}}_M^d < \infty$.

b) Suppose α is a nonzero algebraic integer all of whose conjugates lie on or inside the unit circle. Show that α is a root of unity.