Homework 10 - Number Theory

Exercise 1. Let p be a prime number.

a) The rationals $\mathbb{Q}$ is dense in the p-adic numbers $\mathbb{Q}_p$.

b) The integers $\mathbb{Z}$ is dense in the p-adic integers $\mathbb{Z}_p$.

c) (Weak Approximation.) Suppose K a field with pairwise inequivalent nontrivial valuations $|\cdot|_1, \ldots, |\cdot|_n$ (a.k.a. absolute values). Show that K is dense in $\prod \widehat{K}_i$.

Exercise 2. (Local-to-global.)

a) Show that $x \in \mathbb{Q}$ is a square if and only if it is a square in all completions $\mathbb{Q}_v$ for $v \in M_{\mathbb{Q}}$.

b) The equation

$$(X^2 - 2)(X^2 - 14)(X^2 - 34) = 0$$

has a zero in $\mathbb{Q}_v$ for $v \in M_{\mathbb{Q}}$, but no zero in $\mathbb{Q}$.

Exercise 3. (Structure of the multiplicative group $\mathbb{Q}_2^\times$.)

a) Show there exist isomorphisms

$$U^{(1)} \cong \mathbb{Z}/2 \times \mathbb{Z}_2 \text{ and } U^{(2)} \cong \mathbb{Z}_2.$$

Conclude that $\mathbb{Q}_2^\times$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}/2 \times \mathbb{Z}_2$.

b) The classes of 2, -1 and 5 generate the group of square classes

$$\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2 \cong (\mathbb{Z}/2)^3.$$

c) Tabulate the distinct quadratic extensions of $\mathbb{Q}_2$.

Exercise 4. (Continuity of roots.) Let K_v be a local number field and $\alpha, \beta \in \overline{K_v}$. Then we say that β belongs to α if $|\alpha - \beta| < |\sigma(\alpha) - \alpha|$ for every $\sigma \in \mathrm{Gal}(\overline{K_v}/K_v)$ for which $\sigma(\alpha) \neq \alpha$. For $h(X) = a_n X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in K_v[X]$ define $|h| = \max\{|a_i|\}$.

Now suppose $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in K_v[X]$ is irreducible and $\alpha \in \overline{K_v}$ is a zero of f. There exists a constant $c(f) > 0$ such that if $g(X) \in K_v[X]$ is any monic polynomial for which $|g - f| \leq c(f)$, then g has a zero β that belongs to α. Moreover, g is irreducible, $\deg(f) = \deg(g)$, and $K_v(\alpha) = K_v(\beta)$.