Homework 11 - Number Theory

Exercise 1. Let K_v be a local number field with residue field k_v of characteristic $p > 0$. In this exercise we show there is a bijection

$$\{L/K_v \text{ finite unramified extension}\} \cong \{l/k_v \text{ finite extension}\}$$

given by

$$L \mapsto l = \mathcal{O}_L/\mathfrak{m}_L$$

with the additional properties

- $L_1 \subseteq L_2$ if and only if $l_1 \subseteq l_2$.
- $\mathrm{Gal}(L/K_v) \cong \mathrm{Gal}(l/k_v)$.

a) When $p \not| m$ the irreducible factors of $X^m - 1$ in $k_v[X]$ are the reductions modulo $\mathfrak{m}_v$ of the irreducible factors of $X^m - 1$ in $\mathcal{O}_v[X]$.

b) If l/k_v has degree n, there exists a unique finite unramified extension K_n/K_v with residue field l.

c) Show that $\mathrm{Gal}(K_n/K_v) \cong \mathrm{Gal}(l/k_v)$.

Exercise 2. Let L_w/K_v be a finite extension of local number fields.

a) Show that L_w/K_v is totally tamely ramified if and only if there exists uniformizers ω_v of K_v and ω_w of L_w such that

$$\omega_v = \omega_v^{[L_w:K_v]},$$

where $p \not| [L_w : K_v]$.

b) There exists unique subfields

$$K_v \subseteq L_w^{ur} \subseteq L_w^t \subseteq L_w,$$

where L_w^{ur} is the maximal unramified extension of K_v contained in L_w, L_w^t/L_w^{ur} is totally tamely ramified, and L_w/L_w^t is totally wildly ramified.

c) Show that $\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p$ is totally ramified.

d) Read §1,2 of Chapter IV in the textbook.

e) Let K_v^{ur} denote the maximal unramified extension of K_v contained in a fixed algebraic closure $\overline{K_v}$. Identify the Galois group $\mathrm{Gal}(K_v^{ur}/K_v)$ with the profinite integers $\widehat{\mathbb{Z}}$.

Exercise 3. (Hilbert symbols.) Let F be a field of characteristic not equal to 2.

a) Show that the following statements are equivalent for units $a, b \in F^\times$.

(i) The equation

$$X^2 - aY^2 - bZ^2 = 0$$

is solvable with $(X, Y, Z) \in F^3$, not all zero.

(ii) The equation

$$X^2 - aY^2 - bZ^2 + abW^2 = 0$$

is solvable with $(X, Y, Z, W) \in F^4$, not all zero.

(iii) The element b is a norm from $F(\sqrt{a})$.

(iv) The element a is a norm from $F(\sqrt{b})$.

The Hilbert symbol $(a,b)_F$ is defined to be 1 if the above equivalent statements are satisfied; otherwise, we define $(a,b)_F = -1$.

b)

- Show that $(a,b)_F = (b,a)_F = (ac^2, bd^2)_F$ for all units $c,d \in F^\times$. Conclude that the Hilbert symbol induced a well-defined map
$$F^\times/(F^\times)^2 \times F^\times/(F^\times)^2 \to \mu_2.$$
- Show the formulas $(a,-a)_F = (a, 1-a)_F = 1$.
- If $(a,b)_F = 1$ then $(aa', b)_F = (a', b)_F$ for all $a' \in F^\times$.