# Homework 13 - Number Theory

**Exercise 1.** (Hilbert symbols, continued.) The following statements are equivalent:

(i) (Legendre-Gauss) For odd relative prime positive integers $a$ and $b$,

$$(\frac{a}{b}) = (-1)^{\frac{a-1}{2}\frac{b-1}{2}}(\frac{b}{a}).$$

(ii) (Hilbert) For all $a, b \in \mathbb{Q}^\times$ the product formula

$$\prod_{v \in M_{\mathbb{Q}}} (a, b)_v = 1$$

holds. The product is taken over all the places of the rationals.

(iii) (Euler-Artin) As a function of $p$, the Legendre symbol $(\frac{a}{p})$ for $a > 0$ depends only on the residue class of $p$ modulo the discriminant $\Delta_a$ of $\mathbb{Q}(\sqrt{a})$.

**Exercise 2.** (The second $K$-group.) Let $F$ be a field. Define $K_2(F)$ to be the free abelian group generated by symbols $\{a, b\}$ for units $a, b \in F^\times$ modulo the subgroup generated by the relations

$$\{aa', b\} = \{a, b\}\{a', b\} \text{ for } a, a', b \in F^\times,$$
$$\{a, bb'\} = \{a, b\}\{a, b'\} \text{ for } a, b, b' \in F^\times,$$
$$\{a, 1-a\} = 1 \text{ for } a \in F^\times \setminus \{1\}.$$

With this definition, the map $\{-,-\}\colon F^\times \times F^\times \to K_2(F)$ is universal for "symbol maps" $F^\times \times F^\times \to G$ taking values in abelian groups.

**a**) Show the relations

$$\{a, -a\} = 1 \text{ for } a \in F^\times,$$
$$\{a, b\} = \{b, a\}^{-1} \text{ for } a, b \in F^\times,$$

**b**) Let $\mathbb{F}$ be a finite field. Then $K_2(\mathbb{F})$ is the trivial group.

**c**) Show that the following are equivalent:

(i) For every $E = F(\sqrt{a})$, $a \in F$, the subgroup $N_{E/F}(E^\times)$ has index 1 or 2 in $F^\times$.

(ii) The Hilbert symbol $(-,-)_F\colon F^\times \times F^\times \to \mu_2$ is bimultiplicative.

(iii) The Hilbert symbol $(-,-)_F\colon F^\times \times F^\times \to \mu_2$ is a symbol map, inducing a group homomorphism $K_2(F) \to \mu_2$ taking $\{a, b\}$ to $(a, b)_F$.

**Exercise 3.** **a**) Read Chapter III of J. Milne's notes on class field theory:

*http://www.jmilne.org/math/CourseNotes/CFT.pdf*

**b**) Prove Proposition 1.8 on pg. 98 of Milne's notes.

**c**) The Kronecker-Weber theorem in Chapter V, §1, of the textbook.