Homework 14 - Number Theory

Exercise 1. (Adeles, ideles.) Let K be a number field.

a) Suppose H is a discrete subgroup of a Hausdorff topological group G. Show that H is closed in G. Deduce that K is a closed additive subgroup of $\mathbb{A}_K$.

b) The $\mathbb{A}_\mathbb{Q}$-subspace topology on $\mathbb{I}_\mathbb{Q}$ does not coincide with the usual topology on the rational ideles.

c) Show that $\mathbb{A}_K$ is a topological ring.

d) Show the equality

$$\mathbb{Q} + ([-\frac{1}{2}, \frac{1}{2}] \times \prod_{p \in \mathrm{Max}(\mathbb{Z})} \mathbb{Z}_p) = \mathbb{A}_\mathbb{Q}.$$

e) The volume map $\mathbb{I}_K \to \mathbb{R}_{>0}$ is continuous.

Exercise 2. a) Identify $\mathbb{I}_\mathbb{Q}$ with

$$\mathbb{Q}^\times \times \mathbb{R} \times \widehat{\mathbb{Z}}^\times$$

and the idele class group $C_\mathbb{Q}$ of the rationals with

$$\mathbb{R} \times \widehat{\mathbb{Z}}^\times.$$

b) Conclude there is an isomorphism

$$C_\mathbb{Q}^0 \cong \widehat{\mathbb{Z}}^\times.$$

Identify $C_\mathbb{Q}^0$ with the Galois group of a certain field extension of the rationals.

Exercise 3. (Tate's computation of the second K-group of the rationals.)

By the universal property of K_2, restricting the tame symbol τ_p for p odd and the Hilbert symbol $(-,-)_2$ to the rationals $\mathbb{Q}$ induce maps

$$\overline{\lambda_p}\colon K_2(\mathbb{Q}) \to R_p$$

for each $p \in \mathrm{Max}(\mathbb{Z})$. Here $R_p = (\mathbb{Z}/p)^\times$ if p is odd, and $R_2 = \mathbb{Z}/2$. Show that

$$\prod_{p \in \mathrm{Max}(\mathbb{Z})} \overline{\lambda_p}\colon K_2(\mathbb{Q}) \to \prod_{p \in \mathrm{Max}(\mathbb{Z})} R_p$$

is an isomorphism.

(Hint: Consider the subgroup Λ_n of $K_2(\mathbb{Q})$ generated by symbols $\{a, b\}$, where a and b are units or primes less or equal to n. With this definition, $K_2(\mathbb{Q})$ is the direct limit of the groups Λ_n. Use induction to show there is an isomorphism

$$\Lambda_p \cong \prod_{2 \leq q \leq p} R_q.)$$