Homework 6 - Number Theory

Exercise 1. **a**) Let $r, c \geq 0$ be integers such that $r + 2c = n \geq 1$ and let $t > 0 \in \mathbb{R}$. Show that the region

$$R_t = \{(x_1, \ldots, x_n) \in \mathbb{R}^n ||x_1| + \cdots + |x_r| + 2\sqrt{x_{r+1}^2 + x_{r+2}^2} + \cdots + 2\sqrt{x_{n-1}^2 + x_n^2} \leq t\}$$

is compact, convex, symmetric about the origin, and of volume $2^r(\frac{\pi}{2})^c \frac{t^n}{n!}$.

b) Let K be a number field of degree n with r real embeddings and c pairs of complex embeddings. Define the map

$$\varphi \colon K \to \mathbb{R}^n$$

by

$$\alpha \mapsto (\sigma_1(x), \ldots, \sigma_{r+c}(x))$$

via the usual identification $\mathbb{C} \cong \mathbb{R}^2$. If $\psi(x) \in R_t$ show that $|N_{K/\mathbb{Q}}(x)| \leq \frac{t^n}{n^n}$.

c) Let R be a bounded region of $\mathbb{R}^n$ and Γ a full lattice with fundamental domain of volume V. If all translates of R under Γ are disjoint, then $\mathrm{vol}(R) \leq V$. The intersection $\Gamma \cap R$ comprises only a finite number of points.

d) Suppose $R \subseteq \mathbb{R}^n$ compact, convex and symmetric about the origin and $\Gamma \subseteq \mathbb{R}^n$ as in **c**). If $\mathrm{vol}(R) \geq 2^n V$, then $(\Gamma \setminus \{0\}) \cap R \neq \emptyset$.

Exercise 2. Dirichlet's unit theorem for orders in number fields.

a) Read §12 in the textbook.

Let R be an order in a number field K. Define the map

$$\Psi \colon K^\times \to \mathbb{R}^{r+c}$$

by

$$\alpha \mapsto (\log|\sigma_1(\alpha)|, \ldots, \log|\sigma_r(\alpha)|, 2\log|\sigma_{r+1}(\alpha)|, \ldots, 2\log|\sigma_{r+c}(\alpha)|).$$

(This is a group homomorphism from the group of units $K^\times$ under multiplication to the additive group $\mathbb{R}^{r+c}$.) Consider the claim:

$$\text{The image } \Psi(R^\times) \subseteq \mathbb{R}^{r+c} \text{ is a lattice of rank } r+c-1, \tag{1}$$
$$\text{spanning the hyperplane } H \colon x_1 + \cdots + x_{r+c} = 0.$$

b) The claim (1) implies

$$R^\times \cong \mu(R) \times \mathbb{Z}^{r+c-1}.$$

c) Show that $\Psi(R^\times)$ is a lattice contained in the hyperplane H.
(Hint: If $A \subseteq \mathbb{R}^n$ is an additive subgroup such that $A \cap B$ is a finite set for every bounded region $B \subseteq \mathbb{R}^n$, then A is a lattice.)

d) For any set of $r + c$ positive real numbers $\lambda = (\lambda_1, \ldots, \lambda_{r+c})$ let P_λ be the region of $\mathbb{R}^n$ comprising elements $(x_1, \ldots, x_n)$ with $|x_i| \leq \lambda_i$ for $1 \leq i \leq r$ and $x_{r+2i-1}^2 + x_{r+2i}^2 \leq \lambda_{r+i}$ for $1 \leq i \leq c$. Show that P_λ is compact, convex and symmetric about the origin, and has volume $2^r \pi^c \prod_{i=1}^{r+c} \lambda_i$.

e) Show that $\Psi(R^\times)$ spans the hyperplane H.
(Hint: There exists a bounded region S of the hyperplane H such that its translates under $\Psi(R^\times)$ cover H.)

Exercise 3. **a**) Read §7 in the textbook.
b) Show the regulator of a number field does not dependent on any choices.
c) Discuss the regulator of a real quadratic number field.

Exercise 4. Let $d > 0$ be a square-free positive integer such that (i) $d \equiv 1, 2 \bmod 4$, (ii) $d \neq 3a^2 \pm 1$ for any integer a, (iii) 3 does not divide the class number of $\mathbb{Q}(\sqrt{-d})$.
a) Give examples of integers satisfying conditions (i)-(iii).
b) The equation $y^2 = x^3 - d$ has no integer solutions.
c) All of the assumptions on d are required for the conclusion in **b**).

Exercise 5. Compute the ideal class groups of $\mathbb{Q}(\sqrt{-14})$ and $\mathbb{Q}(\sqrt{-21})$.