Homework 7 - Number Theory

Please hand in your solution of Exercise 2 in my mailbox on the 7th floor.

Exercise 1. Let $d \geq 1$ be a square-free integer and denote by $h(-d)$ the class number of the totally imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$.

a) If $h(-d) = 1$, then $d = 1, 2$ or $d = p$, where $p \equiv 3 \bmod 4$ is a prime number.

b) Let $p \equiv 3 \bmod 4$ be a prime number. Show that the following are equivalent.

(a) The Legendre symbol $(\frac{-p}{\ell}) = -1$ for every prime number $\ell < \frac{2}{\pi}\sqrt{p}$.
(b) The class number $h(-p) = 1$.
(c) The Legendre symbol $(\frac{-p}{\ell}) = -1$ for every prime number $\ell < \frac{p}{4}$.

c) Suppose $p \geq 7$ and $p \equiv 3 \bmod 4$. Prove that (c) implies

(d) The polynomial $f(X) = X^2 + X + \frac{p+1}{4}$ takes prime number values for $0 \leq X \leq \frac{p-7}{4}$ an integer.
Note that (d) implies
(e) The polynomial $f(X) = X^2 + X + \frac{p+1}{4}$ takes prime number values for $0 \leq X \leq [\frac{1}{2}(\frac{2}{\pi}\sqrt{p} - 1)]$ an integer.

d) Prove that (e) implies (a).

e) The polynomial $f(X) = X^2 + X + 41$ takes prime number values for $0 \leq X \leq 39$ an integer. (This was first noticed by Euler.)

Exercise 2. Give explicit computations of the discriminant, ring of integers, group of units, and class group of your favorite number field F of degree 3 or 4.

Exercise 3. Review §1-§12 in the textbook. Read §13-§14 in order to enhance your understanding of Chapter 1.