Homework 8 - Number Theory

Exercise 1. Let $(K, |\cdot|)$ be a nonarchimedean valued field.
a) If $|x| \neq |y|$ then $|x+y| = \max\{|x|, |y|\}$.
b) Every triangle in K is isosceles (two of the three sides are equal).
c) Every point inside an open ball is a center of the same ball.
d) Any two open balls are either disjoint or contained in one another.

Exercise 2. a) For any prime p there are $p-1$ distinct $(p-1)$th roots of unity in $\mathbb{Z}_p$.
b) Let m be a square-free integer and p a prime such that $p \nmid 2m$. When is m a square in $\mathbb{Z}_p$?
c) Let $N > 1$ be an integer and define $R_N \equiv \lim \mathbb{Z}/N^n$ with respect to the canonical transition maps $\pi_n \colon \mathbb{Z}/N^{n+1} \to \mathbb{Z}/N^n$. Then there is a ring isomorphism

$$R_N \cong \prod_{p|N} \mathbb{Z}_p.$$

Now define

$$\widehat{\mathbb{Z}} \equiv \{(x_m)_{m\geq 1} | x_n \in \mathbb{Z}/n, \pi_{m,n}(x_{mn}) = x_m \text{ for all } m, n\}.$$

for the canonical maps $\pi_{m,n} \colon \mathbb{Z}/mn \to \mathbb{Z}/m$. There is a ring isomorphism

$$R_N \cong \prod_{p \in \mathrm{Max}(\mathbb{Z})} \mathbb{Z}_p.$$

d) For $n \geq 0$ there is a short exact sequence

$$0 \to \mathbb{Z}_p \to \mathbb{Z}_p \to \mathbb{Z}/p^n \to 0.$$

e) Show that 16 is an 8th power in $\mathbb{Q}_p$ for every odd prime p.

Exercise 3. In this exercise we employ the geometry of numbers to show that every positive integer can be written as a sum of four integer squares (theorem attributed of Lagrange). Enjoy!
a) Let p be an odd prime. Show that the congruence $u^2 + v^2 \equiv -1 \bmod p$ has a solution in integers u and v.
b) Fix u and v as in **a**). Let $\Gamma \subseteq \mathbb{R}^4$ consist of all (a, b, c, d) such that $c \equiv ua + vb \bmod p$ and $d \equiv ub - v \bmod p$. Show that Γ is a full lattice in $\mathbb{R}^4$ and compute its covolume.
c) Compute the volume of the sphere of radius r in $\mathbb{R}^4$.
d) Prove that p can be written as a sum of four integer squares.
e) A quaternion is an expression of the form $z = a + bi + cj + dk$ where i, j, k are formal symbols satisfying $i^2 = j^2 = k^2 = -1$ and $ijk = -1$. Its conjugate is given by $\overline{z} = a - bi - cj - dk$. Look up the definition of the non-commutative ring structure on the quaternions on the world wide web. Show that $z\overline{z}$ is a non-negative real number and define the norm of z as $\sqrt{z\overline{z}}$. Show that the norm defined in this way is multiplicative.
f) Use **d**) and **e**) to show Lagrange's theorem.