Homework 9 - Number Theory

Exercise 1. In this exercise we patch the proof of one result presented in the lecture on 10/13/2009.

Let $(K, |\cdot|)$ be a discrete and complete nonarchimedean valued field and L/K a finite separable extension. Then $|\cdot|$ extends uniquely to a discrete and nonarchimedean valuation $|\cdot|_L$ on L such that for all $y \in L$,

$$|y|_L = |N_{L/K}(y)|^{\frac{1}{[L:K]}}$$

Moreover, $(L, |\cdot|_L)$ is a complete field.

The proof can be divided into the following steps.

a) Let $\mathcal{O}_L$ denote the integral closure of $\mathcal{O}_K$ in L. Show that $\mathcal{O}_L$ is a Dedekind domain and a finite free $\mathcal{O}_K$-module.

b) Show that $\mathcal{O}_L$ has a unique maximal ideal.

c) The valuation $|\cdot|$ extends to a discrete valuation on L.

d) For $y \in L$, $|y|_L = |N_{L/K}(y)|^{\frac{1}{[L:K]}}$.

e) The field L is complete with respect to $|\cdot|_L$.

f) If $|\cdot|_{\prime}$ extends the valuation $|\cdot|$ to L and $x \in \mathcal{O}_L$, then $|x|_{\prime} \leq 1$ (with equality if x is a unit). Conclude the $|\cdot|_{\prime} = |\cdot|_L$.

Exercise 2. The purpose of the following is to clarify and finish off the proofs presented in the lecture on 10/13/2009.

a) (Extending valuations and irreducible factors) Suppose $L = K(\alpha)$ is a finite extension of number fields and let $f(X)$ be the minimal polynomial of α in $K[X]$. There is a one-to-one correspondence between the extensions to L of a valuation $|\cdot|$ on K and the irreducible factors of $f(X)$ in $\widehat{K}[X]$.

b) Suppose $|\cdot|$ is a valuation on $K = \mathbb{Q}(\alpha) = \mathbb{Q}[X]/(f)$ that restricts to the p-adic valuation $|\cdot|_p$ on the rationals. Factor $f(X) = f_1(X) \cdots f_r(X)$ into irreducible polynomials in $\mathbb{Q}_p[X]$ and write $K \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_{i=1}^r \widehat{K_i}$, where $\widehat{K_i} = \widehat{K}[X]/(f_i)$. Let $|\cdot|_i$ be the unique extension of $|\cdot|$ to $\widehat{K_i}$ (these give the possible extensions of $|\cdot|_p$ to K). Show that $|\cdot|_i$ is equivalent to $|\cdot|_{\mathfrak{p}}$ for some prime $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_F)$.

Exercise 3. **a**) For a p-adic number $x \in \mathbb{Q}_p$, use the power series representation of x to show there exists a decomposition

$$x = [x] + \langle x \rangle \in \mathbb{Z}_p + \mathbb{Z}[\frac{1}{p}].$$

Can you give a similar decomposition of any rational number?

b) Use **a**) to define the group homomorphism $\phi\colon \mathbb{Q}_p \to \mathbb{C}^\times$, $x \mapsto \exp(2\pi i \langle x \rangle)$, and show the additive group $\mathbb{Q}_p/\mathbb{Z}_p$ is isomorphic to the multiplicative group $\mu_{p^\infty} = \cup_{k \geq 1} \mu_{p^k}$ of all pth power roots of unity contained in $\mathbb{C}$.

c) The only $\mathbb{Q}$-isomorphism of the p-adic numbers $\mathbb{Q}_p$ is the identity map.