Notes 9: Eli Cartan's theorem on maximal tori.

Version 1.00 — still with misprints, hopefully fewer

Tori Let T be a torus of dimension n. This means that there is an isomorphism $T \simeq \mathbb{S}^1 \times \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$. Recall that the Lie algebra $\operatorname{Lie} T$ is trivial and that the exponential map is a group homomorphism, so there is an exact sequence of Lie groups

$$0 \longrightarrow N \longrightarrow \operatorname{Lie} T \xrightarrow{\exp_T} T \longrightarrow 1$$

where the kernel N_T of $\exp_T$ is a discrete subgroup $\operatorname{Lie} T$ called the *integral lattice* of T.

The isomorphism $T \simeq \mathbb{S}^1 \times \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ induces an isomorphism $\operatorname{Lie} T \simeq \mathbb{R}^n$ under which the exponential map takes the form

$$\exp(t_1, \ldots, t_n) = (e^{2\pi i t_1}, e^{2\pi i t_2}, \ldots, e^{2\pi i t_n}).$$

Irreducible characters. Any irreducible representation V of T is one dimensional and hence it is given by a multiplicative character; that is a group homomorphism $\chi\colon T \to \operatorname{Aut}(V) = \mathbb{C}^*$. It takes values in the unit circle $\mathbb{S}^1$ — the circle being the only compact and connected subgroup of $\mathbb{C}^*$ — hence we may regard χ as a Lie group map $\chi\colon T \to \mathbb{S}^1$.

The character χ has a derivative at the unit which is a map $\theta = d_e\chi : \operatorname{Lie} T \to \operatorname{Lie} \mathbb{S}^1$ of Lie algebras. The two Lie algebras being trivial and $\operatorname{Lie} \mathbb{S}^1$ being of dimension one, this is just a linear functional on $\operatorname{Lie} T$. The tangent space $T_e\mathbb{S}^1 \subseteq T_e\mathbb{C}^* = \mathbb{C}$ equals the imaginary axis $i\mathbb{R}$, which we often will identify with $\mathbb{R}$. The derivative θ fits into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N_T \arrow[r] \arrow[d, "\theta|_{N_T}"] & \operatorname{Lie} T \arrow[r, "\exp"] \arrow[d, "\theta"] & T \arrow[r] \arrow[d, "\chi"] & 1 \\
0 \arrow[r] & N_{\mathbb{S}^1} \arrow[r] & \operatorname{Lie} \mathbb{S}^1 \arrow[r, "\exp"] & \mathbb{S}^1 \arrow[r] & 1
\end{tikzcd}$$

The linear functional θ is not any linear functional, the values it takes on the discrete subgroup N_T are all in $N_{\mathbb{S}^1}$. Choosing $2\pi i \in \operatorname{Lie} \mathbb{S}^1$ as the basis, the elements of $N_{\mathbb{S}^1}$

will be just the integers, and we may phrase the behavior of θ as taking integral values on the integral lattice.

On the other hand, any such linear functional passes to the quotients and induces a group homomorphisms $T \to \mathbb{S}^1$, that is, a multiplicative character on T. Hence we have

Proposition 1 *Let T be a torus. The irreducible characteres of T correspond to the linear functionals on* $\operatorname{Lie} T$ *taking integral values on N_T.*

In the case $T = \mathbb{S}^1 \times \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$, the discrete subgroup N_T is, when we use the basis $2\pi i$ for $\operatorname{Lie}\mathbb{S}^1$, just the subgroup $N_T = \mathbb{Z}^n \subseteq \mathbb{R}^n$ of points having integral coordinates, and a linear functional $c(t_1, \ldots, t_n) = \sum \alpha_i t_i$ takes integral values on N_T if and only if all the coefficients α_i are integral.

REAL REPRESENTATIONS. Let T be a torus, and let V be a *real irreducible* representation of T, *i.e.,* V is a real vector space and the representation is given by $\rho\colon T \to \operatorname{Aut}_{\mathbb{R}}(V)$.

Lemma 1 *Assume that V is a real, non trivial-representation of T, then* $\dim_{\mathbb{R}} V = 2$*, and there is basis for V and a character χ of T such that*

$$\rho(t) = \begin{pmatrix} \operatorname{Re}\chi(t) & \operatorname{Im}\chi(t) \\ -\operatorname{Im}\chi(t) & \operatorname{Re}\chi(t) \end{pmatrix}$$

PROOF: This is standard. Let $W = V \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of V.It has the natural conjugation map $w \mapsto \overline{w}$ defined as $v \otimes z \mapsto v \otimes \overline{z}$. Then $V = V \otimes 1 \subseteq W$ is the subspace of *real* vectors characterized by $v = \overline{v}$. Furthermore the induces representation $\rho \otimes \mathrm{id}_{\mathbb{C}}\colon T \to \operatorname{Aut}_{\mathbb{C}}(V)$ — which we still will denote ρ — is real, *i.e.,* $\overline{\rho(t) w} = \rho(t)\overline{w}$.

Let $w \in W$ be an eigenvector of T with character χ. Then the representation being real, $\overline{w}$ is an eigenvector with $\overline{\chi}$ as character, and if $\chi \neq \overline{\chi}$, the two are linearly independent. If $\chi = \overline{\chi}$, then χ is real, hence $\chi(t) = 1$ for all $t \in T$, and V is the trivial representation. So we assume that χ is not real.

With this assumption, the two vectors $v_+ = w + \overline{w}$ and $v_- = i(w - \overline{w})$ are real and linearly independent. One checks that

$$\rho(t)v_+ = \chi\, w + \overline{\chi}\, \overline{w} = \operatorname{Re}\chi\, (w + \overline{w}) + \operatorname{Im}\chi\, i(w - \overline{w}) = \operatorname{Re}\chi\, v_+ + \operatorname{Im}\chi\, v_-$$

and

$$\rho(t)v_- = \chi\, iw - \overline{\chi}\, i\overline{w} = -\operatorname{Im}\chi(w + \overline{w}) + \operatorname{Re}\chi i(w - \overline{w}) = -\operatorname{Im}\chi\, v_+ + \operatorname{Re}\chi\, v_-$$

Hence v_+ and v_1 form a basis for an invariant subspace of V of dimension two, which must be the whole of V since V was assumed to be irreducible, and the matrix of χ in this basis is as stated. ❑

Of course, if θ is a linear functional on the Lie algebra $\operatorname{Lie} T$ taking integral values on the integral lattice and is such that $\chi(t) = e^{2\pi i\theta(t)}$, *e.g.*, $\theta = (2\pi i)^{-1} d_e\chi$, the matrix takes the form

$$\rho(t) = \begin{pmatrix} \cos 2\pi\theta(t) & \sin 2\pi\theta(t) \\ -\sin 2\pi\theta(t) & \cos 2\pi\theta(t) \end{pmatrix}.$$

TOPOLOGICAL GENERATORS. Recall that en element $t \in T$ is called a *topological generator* for T if the group $\langle\, t \,\rangle$ consisting of the powers of t is dense in T. *i.e.*, if $\overline{\langle\, t \,\rangle} = T$.

In the one dimensional case, a closed subgroup of $\mathbb{S}^1$ being either finite or the whole circle, we see that the powers of $e^{2\pi i t}$ are dense if and only if t is not a rational number. This has a straight forward generalization to tori of any dimension, normally contributed to Kronecker:

Proposition 2 *Let $v = (t_1, \dots, t_n) \in \mathbb{R}^n$. Then $\exp v$ is a topological generator for $T = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ if and only if $1, t_1, \dots, t_n$ are linearly independent over the rationals.*

PROOF: Let H denote the closure of the group $\langle\, t \,\rangle$ generated by $t = \exp v$. Assume that H is not the whole of T. Then he quotient T/H being a compact, non-trivial, abelian Lie group, is a torus of positive dimension and has a non-trivial character.

If χ is the composition of that character with the projection $T \to T/H$, the map χ is a non-trivial character on T taking the value one on H. But the derivative $\theta = d_e\chi$ is a linear functional $\theta(u_1, \dots, u_n) = \sum \alpha_i u_i$ with the α_i's being integers, since it takes integral values on the integral lattice. Now χ takes the value one on $\operatorname{Lie} H$, and we obtain $\theta(v) = \sum \alpha_i t_i \in Z$ as $v \in H$. This shows that the t_i's are linearly dependent over $\mathbb{Q}$.

On the other hand, if such a linear relation $c(v) = 1$ exists over $\mathbb{Q}$, for some integer β, the functional $\theta = \beta c$ has integral coefficients and satisfies $\theta(v) \in \mathbb{Z}$. Then θ defines a non-trivial character χ whose kernel is a proper subgroup of T containing H. ❑

Corollary 1 *The topological generators are dense in T.*

PROOF: We use induction on n, and we have seen that for $n = 1$, the element t_1 has to avoid $\mathbb{Q}$. In general if $(t_1, .., t_{n-1})$ is linearly independent over $\mathbb{Q}$, there is only a countable number of ways to chose of bad t_n. ❑

It is also worthwhile recalling that $\mathrm{Aut}(T) \simeq \mathrm{Gl}(2, \mathbb{Z})$. This implies that the automorphisms of T are *rigid* in the sense that if ϕ_t is a family of automorphisms *a priori* depending continuously on the parameter t from a connected parameter space (*e.g.,* $\mathbb{R}$), then in fact, it does not depend on t. That is $\phi_t = \phi_{t'}$ for all parameter values t and t'.

Maximal tori. As usual G denotes a compact, connected Lie group. A torus $T \subseteq G$ is called *maximal torus* if it maximal among the subtori of G ordered by inclusion; that is if $T' \subseteq G$ is another torus containing T, then $T' = T$.

Lemma 2 *In any compact, connected Lie group, there are maximal tori.*

PROOF: There are tori in every group, *e.g.,* $\{e\}$. Let $T_1 \subseteq T_2 \subseteq \ldots \subseteq T_i \subseteq \ldots$ be an ascending sequence of tori. The corresponding ascending sequence of the dimensions $\dim T_i$ is bounded by $\dim G$, hence eventually constant, which implies that the sequence of tori is eventually constant. ❑

Before proceeding with the theory, we describe a maximal torus for each of the four classes of the classical groups. It is not difficult to see that in each of these cases, any maximal torus is conjugate to the one we exhibit, illustrating a general and fundamental phenomena for compact groups, that we'll prove in later on.

EXAMPLE 1. — $\mathrm{U}(n)$. The subgroup T of the unitary group $\mathrm{U}(n)$ whose members are all the diagonal matrices

$$\begin{pmatrix} e^{it_1} & 0 & \ldots & 0 \\ 0 & e^{it_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & e^{it_n} \end{pmatrix}$$

with $t_i \in \mathbb{R}$, clearly is a torus of dimension n. It is maximal, since if a matrix g commutes with all of T, it has the same eigenvectors as T, hence is diagonal. For the case of the special unitary group $\mathrm{SU}(n)$, we get a maximal torus by imposing the condition that the determinant be one, *i.e.,* that $\sum_{i=1}^{m} t_i = 0$. ❄

EXAMPLE 2. — SO(n) WITH n EVEN. Assume $n = 2m$. A maximal torus is in this case the subgroup T of matrices of the form

$$\begin{pmatrix} \cos t_1 & \sin t_1 & 0 & \dots & 0 \\ -\sin t_1 & \cos t_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & \cos t_m & \sin t_m \\ 0 & \dots & 0 & -\sin t_m & \cos t_m \end{pmatrix} \tag{$\maltese$}$$

Clearly T is a torus. To see that it indeed is maximal, let $\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_m$ be the decomposition of $\mathbb{R}^n$ into mutually orthogonal two dimensional subspaces corresponding to the blocks in the matrix. This decomposition is in fact the decomposition of $\mathbb{R}^n$ into irreducible T-modules, and the summands are mutually nonisomorphic since their characters are different.

If g is an element in SO(n) commuting with the whole of T, it is a T-module homomorphism, and respects thence the decomposition. The restriction $g|_{V_i}$ of g to any factor V_i is an orthogonal map, and therefore lies in the orthogonal group O(2) of that factor. This group has two components, SO(2) and $r\,$SO(2) where r is any reflection. As $g|_{V_i}$ commutes with the whole of SO(2), it lies in SO(2). This means that g has a block structure like the one in $\maltese$. ❈

EXAMPLE 3. — SO(n) WITH n ODD. In this case, with $n = 2m + 1$, a maximal torus is constituted of the matrices of the form

$$\begin{pmatrix} \cos t_1 & \sin t_1 & 0 & \dots & 0 & 0 \\ -\sin t_1 & \cos t_1 & 0 & \dots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ 0 & \dots & 0 & \cos t_n & \sin t_m & 0 \\ 0 & \dots & 0 & -\sin t_n & \cos t_m & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix}$$

Indeed, any g in SO(n) commuting with the elements in T, must share the eigenvector v corresponding to the eigenvalue one with the members of T. The orthogonal complement $v^\perp$ is invariant under both the torus T and the element g, and we are reduced to the even case. ❈

EXAMPLE 4. — Sp($2m$). Recall that the compact, symplectic group Sp($2m$) consists of the *unitary* $2m \times 2m$-matrices which are of the form

$$\begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix} \tag{✣}$$

where $a, b \in \mathrm{Gl}(m, \mathbb{C})$. It has the subtorus T whose members are the matrices on the form

$$\begin{pmatrix} e^{it_1} & \dots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & 0 & \vdots & \dots & \vdots \\ 0 & \dots & e^{it_m} & 0 & \dots & 0 \\ 0 & \dots & 0 & e^{-it_1} & \dots & 0 \\ 0 & \dots & 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & 0 & \dots & e^{-it_m} \end{pmatrix}$$

This is clearly a m-dimensional torus, and it is maximal. Indeed, any unitary matrix commuting with all its elements must have the same eigenvectors, hence is diagonal, and being of the form ✣, it belongs to T. ❄

THE ADJOINT REPRESENTATION A torus T in G acts via the adjoint representation $\mathrm{Ad}_\star$ on $T_e = \mathrm{Lie}\, G$.

Proposition 3 *Under the adjoint action, the Lie algebra* $\mathrm{Lie}\, G$ *decomposes as a* T*-module as*

$$\mathrm{Lie}\, G = V_0 \oplus \bigoplus_{i=1}^{r} V_{\theta_i}$$

where each $\theta_i \colon \mathrm{Lie}\, T \to \mathbb{R}$ *is a non-trivial linear functional taking integral values on the integral lattice* N_T*, and where* T *acts trivially on* V_0*.*

The torus T *is maximal if and only of* $V_0 = \mathrm{Lie}\, T$*.*

PROOF: The first statement is just the description in lemma 1 of the real representations of T . Let us prove the second. Clearly $\mathrm{Lie}\, T \subseteq V_0$.

If $T \subseteq T'$, then $\mathrm{Lie}\, T \subseteq \mathrm{Lie}\, T' \subseteq V_0' \subseteq V_0$, where V_0' is the part of $\mathrm{Lie}\, G$ where T' acts trivially. If $T \neq T'$, it follows that $\mathrm{Lie}\, T \neq \mathrm{Lie}\, T'$ and hence $\mathrm{Lie}\, T \neq V_0$.

Assume that $\mathrm{Lie}\, T \neq V_0$ and pick a vector v i V_0, but not in $\mathrm{Lie}\, T$. The one-parameter subgroup $H = \{\exp uv \mid u \in \mathbb{R}\}$ is invariant under conjugation by T, indeed

$$x(\exp uv)x^{-1} = \exp \mathrm{Ad}_x uv = \exp uv$$

since $\mathrm{Ad}_x v = v$. It follows that the closure of $H \cdot T$ is a compact, connected and abelian group, hence a torus containing T. ❑

An immediate corollary is:

Corollary 2 $\dim G - \dim T$ *is even.*

THE NORMALIZER OF T AND THE WEYL GROUP Let for a while $T \subseteq G$ be any torus contained in G. Recall from the theory of groups that the *normalizer* of T is the subgroup $N_G T = \{ g \in | \, gTg^{-1} \subseteq T \}$.

This a closed subgroup. Indeed, if t is a topological generator for T, and $\{g_i\}$ is a sequence from $N_G T$ converging to g, then all the $g_i t g_i^{-1}$ are in T, and therefore $gtg^{-1} \in T$ since T is closed. It follows that $gTg^{-1} \subseteq T$, the virtue of t being it is a topological generator.

The torus it self is a closed and normal subgroup of $N_G T$. In fact the normalizer is the largest group containing T in which T is normal. In the case T is maximal, the quotient $W = N_G(T)/T$ does not depend on T, as we shall see, and it is called the *Weyl group of* G. The most important property it has is

Theorem 1 *If* $T \subseteq G$ *is a maximal torus, then the Weyl group* $W = N_G T/T$ *is finite*

PROOF: This is direct consequence of the ridgity of the automorphisms of T. The normalizer $N_G T$ acts on T by conjugation in a continuous way as seen from the commutative diagram

$$\begin{tikzcd} G \times G \arrow[dr] & \\ G \times N_G T \arrow[u] \arrow[r] & G \\ T \times N_G T \arrow[u] \arrow[r] & T \arrow[u] \end{tikzcd}$$

where the maps either are conjugation maps or inclusions. The one we want to be continuous is the bottom one, and all the others are continuous. Hence the action is given by a continuous homomorphism $\psi \colon N_G T \to \mathrm{Aut}(T) \simeq \mathrm{Gl}(n, \mathbb{Z})$.

Now, let N_0 be the connected component of $N_G T$. We claim that $N_0 = T$. The theorem follows then, since $N_G T/N_0$ is compact and discrete, hence finite.

The image $\psi N_0 \subseteq \mathrm{Gl}(n, \mathbb{Z})$ is connected, and it is therefore reduced to the identity element, $\mathrm{Gl}(n, \mathbb{Z})$ being discrete. It follows that all elements in N_0 commute with T.

Let H be the image of any one-parameter-group in N_0. Then $H \cdot T$ is a commutative, connected group, hence a torus containing T, and therefore $H \cdot T = T$ and $H \subseteq T$ as T is a maximal torus. It follows that $N_0 \subseteq T$ since an open neigbourhood of the unit element in N_0 is covered by one-parameter-groups. ❑

Eli Cartan's theorem on maximal tori The fundamental results that underlies the whole theory of representations of compact Lie groups is the one we shall treat in this paragraph. Is goes back to Eli Cartan. The theorem appears in any text on representation theory of compact Lie groups, and there is a multitude of proofs around. The one we present, was found by André Weil and depends on a fixed point theorem from algebraic topology.

Eli Cartan's theorem has two parts, the first stating that given a maximal torus $T \subseteq G$, the different conjugates $g^{-1}Tg$ cover G. That is, any element x is contained in a conjugate of T.

The second part says that any two maximal tori are conjugate, so as far as representation theory goes, one is as good as another.

Theorem 2 *Let G be a compact, connected Lie group. Assume that T is a maximal torus and let $g \in G$ be any element. Then g is contained in a conjugate of T.*

Corollary 3 *Any two maximal tori in a compact, connected Lie group G are conjugate. That is, if T, T' are two such, then $T = x^{-1}T'x$ for some x in G.*

PROOF OF THE COROLLARY.: Chose any generator $t \in T$. By the theorem t is conjugate to some element in T', that is $x^{-1}tx \in T'$ for a suitable element $x \in G$. But as conjugation is a continuous group isomorphism, it maps the closure of the group $\langle\, t \,\rangle$ generated by t, to the closure of $\langle\, x^{-1}tx \,\rangle$, and as the closure of $\langle\, t \,\rangle$ equals T and that of $\langle\, x^{-1}tx \,\rangle$ equals T', we are done. ❑

The proof we shall present of Eli Cartan's theorem, is the one discovered by Andre Weil, and for clarity we split it in several lemmas. The main idea is to search for fixed points of the map

$$l_g \colon G/T \to GT$$

which we know is a diffeomorpism of the compact manifold G/T. The motivation for this search is

Lemma 3 *A coset xT is a fixed point for l_g if and only if $g \in x^{-1}Tx$.*

PROOF: We have $gxT = xT$ if and only if $x^{-1}gxT = T$ which occurs exactly when $x^{-1}gx \in T$. ❑

Several tools for exhibiting fixed points are available in the setting of algebraic topology, but there is one underlying principle. The number of fixed points — at least if they are finite in number, and they must counted in the right way — is a topological quantity, only depending on the homotopy class of the map.

As usual the counting takes orientations into account. If $f\colon X \to X$ is our map, which we assume is a diffeomorphism having finitely many fixed points, and $x \in X$ is a fixed point of f , then the derivative $d_x f$ maps the tangent space $T_x X$ into itself. Hence $\det(\mathrm{id} - d_x F)$ has meaning, and we assume that it is non-zero in any of the fixed points. Then the fixed point x contribute to the number with 1 if $\det(\mathrm{id} - d_x F) > 0$ and with -1 if $\det(\mathrm{id} - d_x F) < 0$. The correct count of fixed points is then $N(f) = \sum_x \operatorname{sign}(\det(\mathrm{id} - d_x F))$.

Since G is connected, left translation $l_h\colon G/T \to G/T$ by any other element h in G is homotopic to l_g, and once we exhibit an h with $N(h) \neq 0$, we are through. It appears that the appropriate choice for h is a topological generator t for T. Certainly it has at least one fixed point, as does any element in T, namely the coset T, but we need to control all the fixed points to do the counting in the proper way:

Lemma 4 *Let $t \in T$ be a generator for T. Then a coset xT is a fixed point for the left translation $l_t\colon G/T \to G/T$ if and only if x belongs to the normalizer $N_G T$. The fixed points of the left translation l_h are finite in number.*

PROOF: This is clear: $txT = xT$ if and only if $x^{-1}tx \in T$, but, t being a topological generator for T, this happens if and only if $x^{-1}Tx = T$, that is when and only when x normalizes T. The cosets fixed by l_x are therefore those in the Weyl group $W = N_G T/T$, and they are finite in number by theorem 1 on page 7. ❑

It remains to determine the contribution of each of the fixed points, and this leads us to computing the derivative of l_h at the fixed points. Luckily, they all behave in the same way:

Lemma 5 *Let $n \in N_G T$ be an element that normalizes T. Then*

$$\det(\mathrm{id} - d_e l_h) = \det(\mathrm{id} - d_n l_h)$$

PROOF: Indeed, if n normalizes T, then $Tn = nT$. Since $xTn = xnT$, the right translation $r_n\colon G/T \to G/T$ is well defined, it takes T to nT, and it commutes with any left action. The following diagram, where we by slight abuse of language refer to the coset nT as n to simplify the notation, is therefore commutative

$$\begin{tikzcd} T_eG/T \arrow[r, "d_e r_n"] \arrow[d, "d_e l_h"'] & T_nG/T \arrow[d, "d_n l_h"] \\ T_eG/T \arrow[r, "d_e r_n"'] & T_nG/T \end{tikzcd}$$

and this shows that $d_e l_h$ and $d_n l_h$ are conjugate maps, hence $\det(\mathrm{id} - d_e l_h) = \det(\mathrm{id} - d_n l_h)$. ❑

To prepare the ground for the final computation at the unit element, recall that by lemma 1 $\mathrm{Lie}\, G$ can decomposed under the adjoint action as a direct sum of irreducible T-modules:

$$\mathrm{Lie}\, G = \mathrm{Lie}\, H \oplus \bigoplus_i V_i.$$

On each component V_i the action of an element $t \in T$ is, in an appropriate basis, given by the matrix

$$\begin{pmatrix} \cos 2\pi\theta_i(t) & \sin 2\pi\theta_i(t) \\ -\sin 2\pi\theta_i(t) & \cos 2\pi\theta_i(t) \end{pmatrix}$$

where the θ_i's are non-trivial linear functionals on $\mathrm{Lie}\, T$ taking integral values on the integral lattice of T.

Lemma 6 *For any element $t \in T$ we have the identity*

$$\det(\mathrm{id} - d_e l_t) = \prod_i (2 - 2\cos 2\pi\theta_i(t)).$$

If t is a topological generator, then $\det(\mathrm{id} - d_e l_t) > 0$.

PROOF: The left translation $l_t\colon G/T \to G/T$ lifts to the conjugation map $c_{t^{-1}}\colon G \to G$ given by $c_{t^{-1}}(x) = txt^{-1}$, indeed $txT = txt^{-1}T$. As $d_e c_{t^{-1}} = \mathrm{Ad}_{t^{-1}}$, we therefore have the commutative diagram of T-equivariant maps which is the key to the proof

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Lie}\, T \arrow[r] \arrow[d, "\mathrm{Ad}_t"] & \mathrm{Lie}\, G \arrow[r] \arrow[d, "\mathrm{Ad}_t"] & T_eG/T \arrow[r] \arrow[d, "d_e l_t"] & 0 \\ 0 \arrow[r] & \mathrm{Lie}\, T \arrow[r] & \mathrm{Lie}\, G \arrow[r] & T_eG/T \arrow[r] & 0 \end{tikzcd}$$

It follows that as a T-module $T_eG/T = \bigoplus_i V_i$. On each component V_i, the action of $\mathrm{id} - \mathrm{Ad}_{t^{-1}}$ is given by the matrix

$$\begin{pmatrix} 1-\cos 2\pi\theta_i(t) & \sin 2\pi\theta_i(t) \\ -\sin 2\pi\theta_i(t) & 1-\cos 2\pi\theta_i(t) \end{pmatrix}$$

whose determinant is $2-2\cos 2\pi\theta_i(h)$. We have proved that $\det(\mathrm{id}-\mathrm{Ad}_h) = \prod_i(2-2\cos 2\pi\theta_i(t))$.

Obviously $2-2\cos 2\pi\theta_i(t) \geq 0$. Now $\theta_i(t^n) \equiv n\theta_i(t)$ mod the integral lattice, so if $2\pi\theta_i(t)$ is an integral multiple of π, the same holds for $\theta_i(x)$ for any $x \in T$, the element t being a generator. This is impossible since the character is non trivial, so $2-2\cos 2\pi\theta_i(t) > 0$. ❑

The classical groups Recall that an $n\times n$ *permutation matrix* is an $n\times n$-matrix with just one one in each column (or row) and the rest of the entries being zeros. They form a subgroup $\Gamma_n \subseteq \mathrm{Gl}(n, \mathbb{K})$ isomorphich to S_n.

If $v_1, \ldots, v_n$ is a basis for a vector space V, any linear map A that permutes the basis vectors — that is $A(v_i) = v_{\alpha(i)}$ for some permutation α of $[1, n]$ — has a permutation matrix as matrix relative to that basis, and vice versa.

There is also a group G_n of so called *signed* permutations, whose elements are permutations σ of the set

$$\{-n, \ldots, -2, -1, 1, 2, \ldots, n\}$$

satisfying

$$\sigma(-i) = -\sigma(i)$$

for all $i \in [1, n]$. Thus there is an exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z}^n \longrightarrow G_n \longrightarrow S_n \longrightarrow 0$$

where the map $G_n \to S_n$ is the forgetful map forgetting the sign.

One may check that G_n can be realized as the set matrices in $\mathrm{Gl}(n, \mathbb{K})$ whose columns (or rows) have exactly one non zero element and that element being either 1 or -1.

We let G_m^0 be the subgroup of even permutations.

EXAMPLE 5. — THE WEYL GROUP OF $\mathrm{U}(n)$. Recall from example 1 on page 4 that a maximal torus T of the unitary group $\mathrm{U}(n)$ is given as the subgroup of all the matrices diagonal in some orthonormal basis $v_1, \ldots, v_n$ for $\mathbb{C}^n$:

$$D(t) = \begin{pmatrix} e^{it_1} & 0 & \ldots & 0 \\ 0 & e^{it_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \ldots & 0 & e^{it_n} \end{pmatrix}$$

where the t_i's are in $\mathbb{R}$ and $t = (t_1, \ldots, t_n)$. The elements of T are characterized among the elements in $\mathrm{U}(n)$ by having the basis vectors $v_1, \ldots, v_n$ as eigenvectors.

As a matter of notation we let $\chi_i(t) = e^{it_i} = e^{i\theta_i(t)}$.

The Weyl group of $\mathrm{U}(n)$ *is isomorphic to the full symmetric group* S_n

Let α be a permutation of $[1, n]$ and define the linear map $a\colon V \to V$ by $a(v_i) = v_{\alpha^{-1}(i)}$, so that a just permutes the basis vectors according to the permutation α^{-1}. Then $a \in U(n)$, the basis being orthonormal. We claim that a normalizes T, indeed

$$ata^{-1}v_i = atv_{\alpha i} = a\chi_{\alpha i}(t)v_{\alpha i} = \chi_{\alpha i}(t)av_{\alpha i} = \chi_{\alpha i}(t)v_i$$

and each v_i is an eigenvector for $a^{-1}ta$, and $ata^{-1} \in T$. On the other hand, in case an element $b \in \mathrm{U}(n)$ normalizes T, each bv_i is an eigenvector for all T. Indeed

$$tbv_i = bb^{-1}tbv_i = b\chi_i(b^{-1}tb)v_i = b\chi_i(b^{-1}tb)v_i = \chi_i(b^{-1}tb)bv_i$$

This means that $bv_i = u_i v_{\alpha i}$ for some permutation α of $[1, n]$ and some scalars u_i. These scalars must all be of modulus one, b being unitary, and it follows that if p is the permutation matrix corresponding to α, then $b = up$, where $u \in T$ is the matrix having the u_i's along the diagonal.

❄

EXAMPLE 6. — THE WEYL GROUP OF $\mathrm{SO}(2m)$.

Let $n = 2m$, and decompose $V = \mathbb{R}^n = V_1 \oplus \cdots \oplus V_m$ as an orthogonal sum of two dimensional subspaces. Then the set of matrices of the form ✚ in example 2 on page 5 form a maximal torus in $\mathrm{SO}(n)$. Call it T. Let $v_1, \ldots, v_n$ be an orthonormal basis for V with v_{2i-1}, v_{2i} being a basis for V_i.

The Weyl group of $\mathrm{SO}(2m)$ *is the group* G_m^0 *of even, signed permutations .*

For any permutation α of $[1, m]$ let a_α be the orthogonal transformation of V defined on the basis above by $a_\alpha(v_{2i-1}) = v_{2\alpha^{-1}(i)-1}$ and $a_\alpha(v_{2i}) = v_{2\alpha^{-1}(i)}$. One checks easily that $a_\alpha a_\beta = a_{\alpha\beta}$, and the set of the a_α's constitute a subgroup Γ_m of $\mathrm{SO}(2m)$ isomorphic to S_m. This subgroup normalizes T: Since $a_\alpha V_i = V_{\alpha^{-1}(i)}$, the conjugate $a_\alpha t a_{\alpha^{-1}}$ has a block structure like the matrix in ✠ in example 2 on page 5, and on each of the subspaces V_i it acts through the matrix

$$\begin{pmatrix} \cos t_{\alpha(i)} & \sin t_{\alpha(i)} \\ -\sin t_{\alpha(i)} & \cos t_{\alpha(i)} \end{pmatrix}$$

Hence the matrix of ata is just the matrix 2 with the blocks permuted.

However, we can produce another elements in the normalizer. Let the map ι_j be defined by $\iota_j(v_{2j}) = v_{2j}$ and $\iota_j(v_{2j+1}) = -v_{2j+1}$, and $\iota(v_k) = v_k$ for the other basis vectors with $k \neq j$. On the component V_j it acts as the reflection through the line spanned by e_{2i}, and it leaves all the other components untouched. Clearly ι_j is an orthogonal transformation of determinant -1. It normalizes T since the reflections in $\mathrm{O}(2)$ normalize $\mathrm{SO}(2)$. All the ι_j's are involutions, *i.e.*, $\iota_j^2 = 1$, and clearly they commute. They constitute thus a subgroup of $\mathrm{O}(2m)$ isomorphic to $\mathbb{Z}/2\mathbb{Z}^m$. One also easily checks that the symmetric subgroup Γ_m permutes the ι_j's. Hence together they constitute a group isomorphic to G_n.

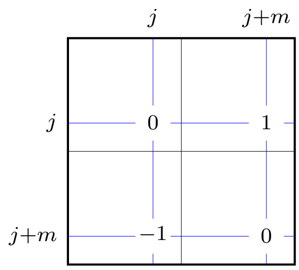
For any even number $2r$, we thus get elements $\iota_1\iota_2 \ldots \iota_{2r}$ in $\mathrm{SO}(2m)$ normalizing T, and this gives us the group G_m^0.

Assume now that a normalizes T. It must permute the factors V_i since aV_i is invariant under T; indeed $taV = a(a^{-1}ta)V = aV$. Call the corresponding permutation of $[1, m]$ for α. Then the element $b = a_\alpha^{-1}a$ in $\mathrm{SO}(2m)$ normalizes T and leaves each of the factors V_i invariant. Being orthogonal, its restriction to V_i is either a rotation or a reflection, so after multiplying b by a suitable element from T, we see that b acts on each factor V_i either as the identity or the reflection

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and we are done. ❋

EXAMPLE 7. — THE WEYL GROUP OF $\mathrm{SO}(2m+1)$. Finally if $n = 2m + 1$, the restriction in the previous example on the number of ι's being even is no longer necessary, but a slight modification of the ι's is needed. They must be defined on the additional basis vector v_{2m+1}, and we do that by letting $\iota_j v_{2m+1} = -v_{2m+1}$. Then they are all of determinant one, and still they are commuting involutions being permuted by Γ_m. Hence



Figur 1: The placement of the 2×2 symplectic matrix in ι_j

The Weyl group of $\mathrm{SO}(2m+1)$ *is* G_m.

❄

EXAMPLE 8. — THE WEYL GROUP OF $\mathrm{Sp}(2m)$. We shall check the following:

The Weyl group of $\mathrm{Sp}(2m)$ *is isomorphic to group* G_m *of signed. permutations.*

However we will be leaving most details to the reader. They can be checked analogous to what we did in the previous examples. We refer to example 4 on page 5, and let $v_1, \ldots, v_{2m}$ be the basis of $\mathbb{C}^{2m}$ of eigenvectors for the matrix in example 4. The maximal torus T is the the set of matrices

$$\begin{pmatrix} e^{it_1} & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & 0 & \vdots & \ldots & \vdots \\ 0 & \ldots & e^{it_m} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & e^{-it_1} & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \ddots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & e^{-it_m} \end{pmatrix}$$

with $t = (t_1, \ldots, t_m) \in \mathbb{R}^m$, Let $\mathbb{C}^{2m} = V_1 \oplus \cdots \oplus V_m$ be the decomposition of $V = \mathbb{C}^{2m}$ in two dimensional complex subspaces, each having basis v_i, v_{i+m}.

The permutations of those subspaces constitute a subgroup of $\mathrm{Sp}(2m)$ isomorphic to S_m, normalizing T.

To get hold of the involutions we use a model that is the 2×2 symplectic matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

It is not an involution, but has square equal to $-\mathrm{id}$, and it has for us the important property that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \overline{\lambda} & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix}$$

So placing its entries in the appropriate positions in the matrix, as is illustrated in figure 1, gives us commuting elements ι_j. However they are not involutions, but ι_j^2 is diagonal, so their images in $W = N_{\mathrm{Sp}(2m)}T/T$ are involutions, and therefore their images generate a subgroup of W isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$ on which S_m acts by conjugation. So, the elements from S_m together with the involutions we just defined, generate a subgroup of W isomorphic to G_m. It is left to the reader to verify that that is all of the Weyl group.

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